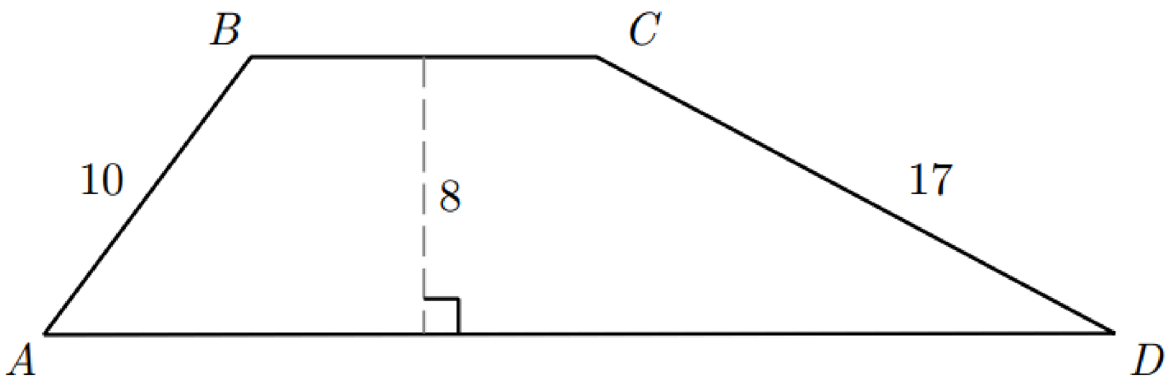
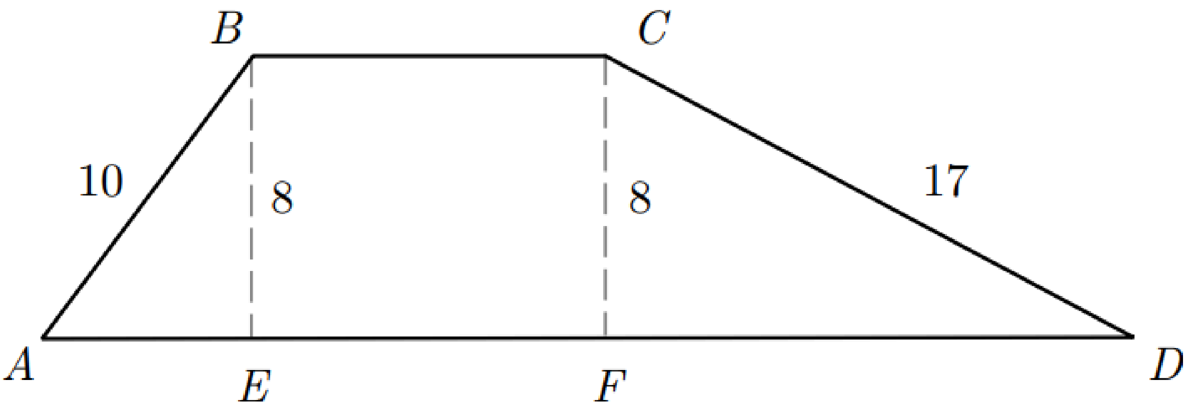
2003 Q21

21. The area of trapezoid $ABCD$ is 164 cm^2. The altitude is 8 cm, AB is 10 cm, and CD is 17 cm. What is BC, in centimeters?



(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 20

21. **(B)** Label the feet of the altitudes from B and C as E and F respectively. Considering right triangles AEB and DFC, $AE = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$ cm, and $FD = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$ cm. So the area of $\triangle AEB$ is $\frac{1}{2}(6)(8) = 24\text{ cm}^2$, and the area of $\triangle DFC$ is $\left(\frac{1}{2}\right)(15)(8) = 60\text{ cm}^2$. Rectangle $BCFE$ has area $164 - (24 + 60) = 80\text{ cm}^2$. Because $BE = CF = 8$ cm, it follows that $BC = 10$ cm.



OR

Let $BC = EF = x$. From the first solution we know that $AE = 6$ and $FD = 15$. Therefore, $AD = x + 21$, and the area of the trapezoid $ABCD$ is $(8)\left\{\frac{1}{2}\left[x + (x + 21)\right]\right\} = 164$. So

$$4(2x + 21) = 164,$$
$$2x + 21 = 41,$$
$$2x = 20,$$

and $x = 10$.

2006 Q21

21. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. The aquarium is filled with water to a depth of 37 cm. A rock with volume 1000 cm^3 is then placed in the aquarium and completely submerged. By how many centimeters does the water level rise?

(A) 0.25 **(B)** 0.5 **(C)** 1 **(D)** 1.25 **(E)** 2.5

21. **(A)** Using the volume formula $lwh = V$, the volume of water in the aquarium is $100 \times 40 \times 37 = 148,000\,\text{cm}^3$. When the rock is put in, the water and the rock will occupy a box-shaped region with volume $148,000 + 1000 = 149,000\,\text{cm}^3$. The volume of the water and the rock is $100 \times 40 \times h$, where h is the new height of the water. The new volume $= 4000h = 149,000\,\text{cm}^3$, so the new height is

$$h = \frac{149000}{4000} = 37.25\,\text{cm}.$$

After adding the rock, the water rises $37.25 - 37 = 0.25\,\text{cm}$.

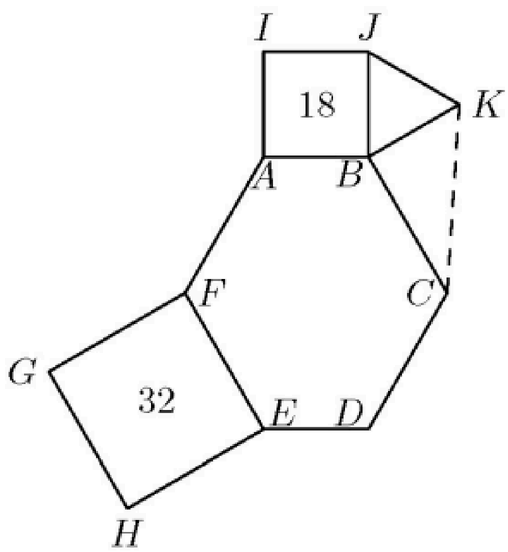
OR

Because the shape of the rock is irrelevant, we may assume that the rock is shaped like a rectangular box with base measuring $100\,\text{cm} \times 40\,\text{cm}$ and height h cm. Using the volume formula, $100 \times 40 \times h = 1000$, so $h = \frac{1000}{100\times 40} = 0.25\,\text{cm}$. When the rock is put into the aquarium, the water level will rise by $0.25\,\text{cm}$.

2015 Q21

21. In the given figure hexagon $ABCDEF$ is equiangular, $ABJI$ and $FEHG$ are squares with areas 18 and 32 respectively, $\triangle JBK$ is equilateral and $FE = BC$. What is the area of $\triangle KBC$?

(A) $6\sqrt{2}$ **(B)** 9 **(C)** 12 **(D)** $9\sqrt{2}$ **(E)** 32



21. **Answer (C):** The area of the square $ABJI$ is 18 and $\triangle KJB$ is equilateral, so $KB = JB = \sqrt{18} = 3\sqrt{2}$. The area of the square $FEHG$ is 32, so $BC = FE = \sqrt{32} = 4\sqrt{2}$. Each interior angle of the hexagon is $120°$, so $\angle KBC = 360° - 60° - 90° - 120° = 90°$ and $\triangle KBC$ is a right triangle. Its area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot 4\sqrt{2} = 12$.

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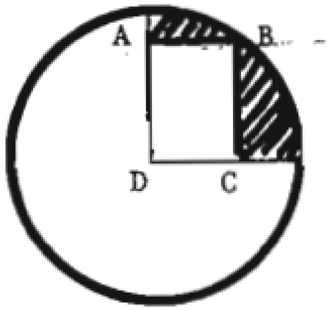
1987 Q22

22. ABCD is a rectangle, D is the center of the circle, and B is on the circle. If AD = 4 and CD = 3, then the area of the shaded region is between



A) 4 and 5 B) 5 and 6 C) 6 and 7

D) 7 and 8 E) 8 and 9

22. D By the Pythagorean Theorem, AC = 5. Since AC = BD, the radius of the circle is 5. The area of the shaded region, then, is $\frac{1}{4} \cdot \pi \cdot 5^2 - 3 \cdot 4$ (a quarter circle with a rectangle deleted) $= \frac{25\pi}{4} - 12$. This quantity is clearly greater than $\frac{76}{4} - 12 = 7$ and less than $\frac{25(3\frac{1}{5})}{4} - 12 = 8$ since $3 < \pi < 3\frac{1}{5}$.

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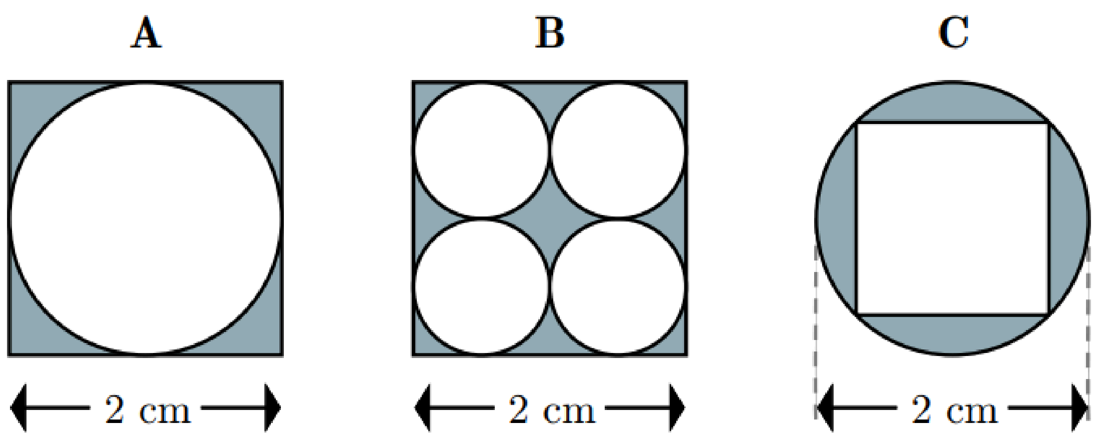
2003 Q22

22. The following figures are composed of squares and circles. Which figure has a shaded region with largest area?



(A) A only **(B)** B only **(C)** C only **(D)** both A and B **(E)** all are equal

22. **(C)** For Figure A, the area of the square is $2^2 = 4$ cm^2. The diameter of the circle is 2 cm, so the radius is 1 cm and the area of the circle is π cm^2. So the area of the shaded region is $4 - \pi$ cm^2.

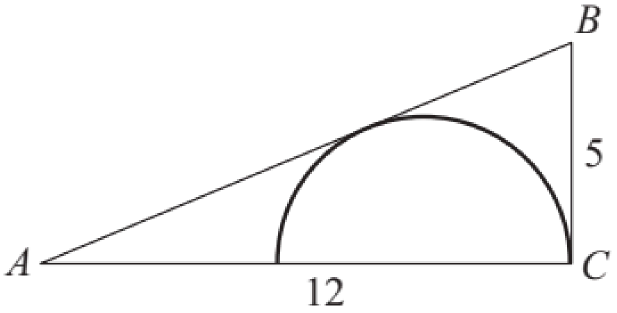
For Figure B, the area of the square is also 4 cm^2. The radius of each of the four circles is $\frac{1}{2}$ cm, and the area of each circle is $\left(\frac{1}{2}\right)^2 \pi = \frac{1}{4}\pi$ cm^2. The combined area of all four circles is π cm^2. So the shaded regions in A and B have the same area.

For Figure C, the radius of the circle is 1 cm, so the area of the circle is π cm^2. Because the diagonal of the inscribed square is the hypotenuse of a right triangle with legs of equal lengths, use the Pythagorean Theorem to determine the length s of one side of the inscribed square. That is, $s^2 + s^2 = 2^2 = 4$. So $s^2 = 2$ cm^2, the area of the square. Therefore, the area of the shaded region is $\pi - 2$ cm^2. Because $\pi - 2 > 1$ and $4 - \pi < 1$, the shaded region in Figure C has the largest area.

Note that the second figure consists of four small copies of the first figure. Because each of the four small squares has sides half the length of the sides of the big square, the area of each of the four small figures is $\frac{1}{4}$ the area of Figure A. Because there are four such small figures in Figure B, the shaded regions in A and B have the same area.

2017 Q22

22. In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



(A) $\frac{7}{6}$ **(B)** $\frac{13}{5}$ **(C)** $\frac{59}{18}$ **(D)** $\frac{10}{3}$ **(E)** $\frac{60}{13}$

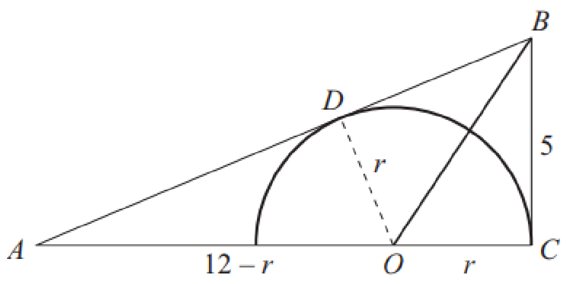
22. **Answer (D):** Let O be the center of the inscribed semicircle on $\overline{AC}$, and let D be the point at which $\overline{AB}$ is tangent to the semicircle. Because $\overline{OD}$ is a radius of the semicircle it is perpendicular to $\overline{AB}$, making $\overline{OD}$ an altitude of $\triangle AOB$. By the Pythagorean Theorem, $AB = 13$. In the diagram, $\overline{OB}$ partitions $\triangle ABC$ so that

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle BOC) + \text{Area}(\triangle AOB)$$

Since we know $\triangle ABC$ has area 30, we have

$$\begin{aligned}30 &= \text{Area}(\triangle BOC) + \text{Area}(\triangle AOB)\\ &= \frac{1}{2}(BC)r + \frac{1}{2}(AB)r = \frac{5}{2}r + \frac{13}{2}r = 9r.\end{aligned}$$

Therefore $r = \frac{30}{9} = \frac{10}{3}$.

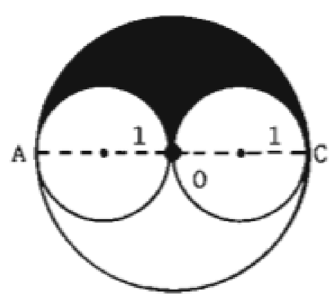


OR

Because $\overline{OD}$ is a radius of the semicircle, it is perpendicular to $\overline{AB}$, making $\triangle ADO$ similar to $\triangle ACB$. Because $\overline{BC}$ and $\overline{BD}$ are both tangent to the semicircle, they are congruent. So $BD = 5$ and $AD = 8$. It follows that $\frac{r}{8} = \frac{5}{12}$ and so $r = \frac{40}{12} = \frac{10}{3}$.

1986 Q23

23. The large circle has diameter AC. The two small circles have their centers on AC and just touch at O, the center of the large circle. If each small circle has radius 1, what is the value of the ratio of the area of the shaded region to the area of one of the small circles?



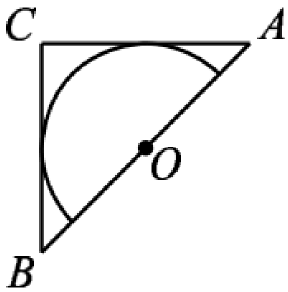
A) between $\frac{1}{2}$ and 1 B) 1 C) between 1 and $\frac{3}{2}$

D) between $\frac{3}{2}$ and 2 E) cannot be determined from the information given

23. (B) The radius of the large circle is 2 since it is a diameter of a small circle. The area of the large circle is $\pi(2)^2 = 4\pi$ and the area of each small circle is π. The shaded area is (by symmetry) half the difference of the areas of the large circle and the two small circles, i.e. $\frac{1}{2}(4\pi - 2\pi) = \pi$. Thus the desired ratio is 1.

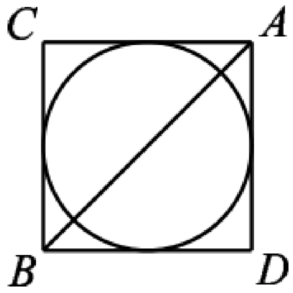
2005 Q23

23. Isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC?



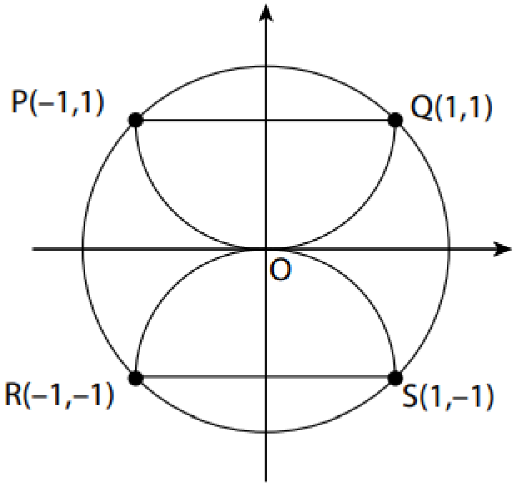
(A) 6 **(B)** 8 **(C)** 3π **(D)** 10 **(E)** 4π

23. **(B)** Reflect the triangle and the semicircle across the hypotenuse $\overline{AB}$ to obtain a circle inscribed in a square. The circle has area 4π. The radius of a circle with area 4π is 2. The side length of the square is 4 and the area of the square is 16. So the area of the triangle is 8.



2010 Q23

23. Semicircles POQ and ROS pass through the center of circle O. What is the ratio of the combined areas of the two semicircles to the area of the circle O?



(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{\pi}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{\sqrt{2}}{2}$

23. **Answer (B):** By the Pythagorean Theorem, the radius OQ of circle O is $\sqrt{2}$. Given the coordinates P, Q, R, and S, the diameters $\overline{PQ}$ and $\overline{RS}$ of the semicircles have length 2. So the areas of the two semicircles will equal the area of a circle of radius 1. Thus the desired ratio is $\dfrac{\pi \cdot 1^2}{\pi(\sqrt{2})^2} = \dfrac{1}{2}$.

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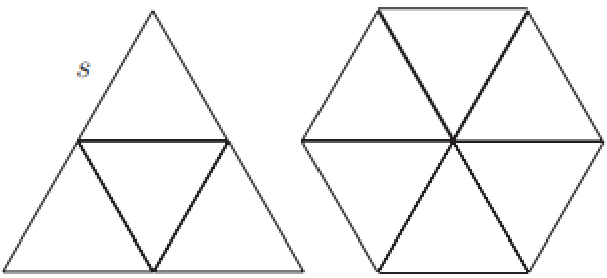
2012 Q23

23. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 4, what is the area of the hexagon?

(A) 4 **(B)** 5 **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** $6\sqrt{3}$

23. **Answer (C):** The equilateral triangle can be divided into 4 smaller congruent equilateral triangles, each with area of 1 and side length s, as shown. Then the original equilateral triangle has perimeter $6s$. A hexagon with perimeter $6s$ can be divided into 6 congruent equilateral triangles, each with area of 1. Therefore the area of the hexagon is 6.



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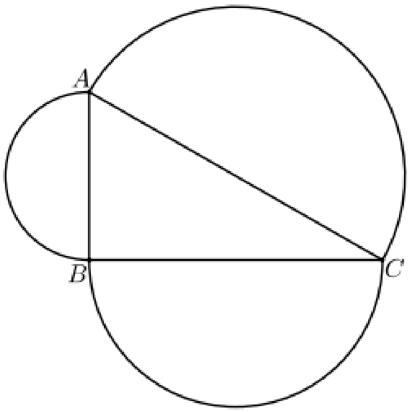
2013 Q23

23. Angle ABC of $\triangle ABC$ is a right angle. The sides of $\triangle ABC$ are the diameters of semicircles as shown. The area of the semicircle on $\overline{AB}$ equals 8π, and the arc of the semicircle on $\overline{AC}$ has length 8.5π. What is the radius of the semicircle on $\overline{BC}$?



(A) 7 **(B)** 7.5 **(C)** 8 **(D)** 8.5 **(E)** 9

23. **Answer (B):** The circle with diameter $\overline{AB}$ has twice the area of the corresponding semicircle; thus the area of the circle is 16π and its radius is 4. Consequently $AB = 8$. The circle with diameter $\overline{AC}$ has circumference 17π, so $AC = 17$. $\overline{AC}$ is the hypotenuse of the right triangle. By the Pythagorean Theorem, $17^2 = 8^2 + (BC)^2$. Therefore $BC = 15$, and the radius is 7.5.

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2016 Q23

23. Two congruent circles centered at points A and B each pass through the other's center. The line containing both A and B is extended to intersect the circles at points C and D. The two circles intersect at two points, one of which is E. What is the degree measure of $\angle CED$?

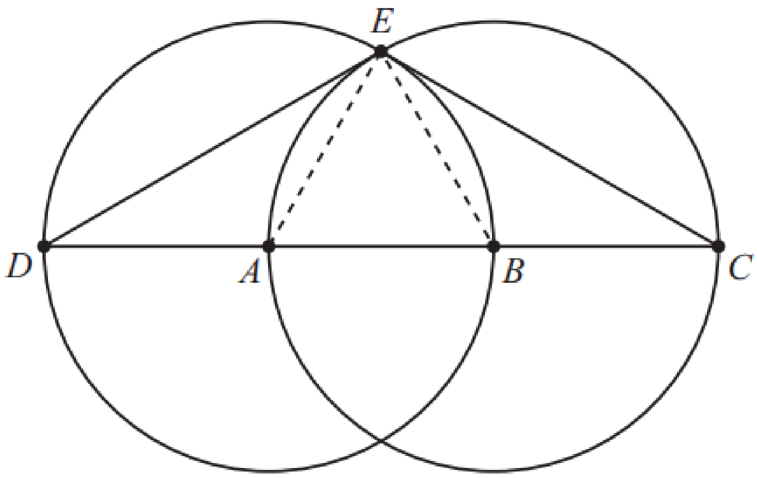
(A) 90 **(B)** 105 **(C)** 120 **(D)** 135 **(E)** 150

23. Answer (C):

We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measure of $\angle AEB$ is 60°. Since $\overline{DB}$ is a diameter of circle A and $\overline{AC}$ is a diameter of circle B, it follows that $\angle DEB$ and $\angle AEC$ are both right angles. Therefore the degree measure of $\angle DEC$ is $90^\circ + 90^\circ - 60^\circ = 120^\circ$.

OR

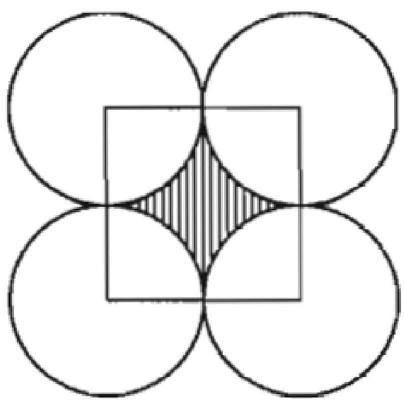
We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measures of $\angle AEB$ and $\angle EAB$ are both 60°. Then the measure of $\angle DAE$ is 120°, and since $\triangle DAE$ is isosceles, the measure of $\angle DEA$ is 30°. Similarly, the measure of $\angle BEC$ is also 30°. Therefore the degree measure of $\angle DEC$ is $30^\circ + 60^\circ + 30^\circ = 120^\circ$.



1992 Q24

24. Four circles of radius 3 are arranged as shown. Their centers are the vertices of a square. The area of the shaded region is closest to

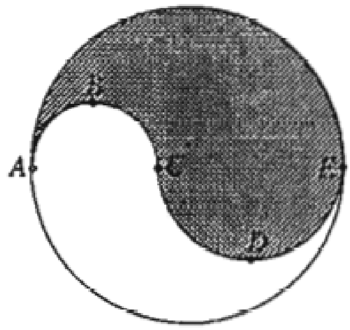
(A) 7.7 **(B)** 12.1 **(C)** 17.2
(D) 18 **(E)** 27



24. **(A)** The four quarter-circles that lie inside the square have a total area equal to the area of one of the circles, 9π. The length of a side of the square is equal to two radii, 6, and thus the square has area 36. The difference is $36 - 9\pi < 36 - 9(3) = 9$, so it is closest to 7.7. (The area, to one decimal place, is 7.7.)

1997 Q24

24. Diameter ACE is divided at C in the ratio 2:3. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. The ratio of the area of the upper region to that of the lower region is

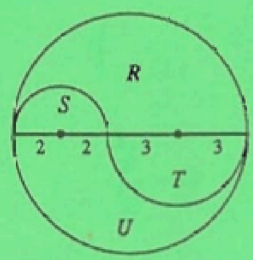


(A) 2:3 **(B)** 1:1 **(C)** 3:2 **(D)** 9:4 **(E)** 5:2

24. **(C)** Let the diameter of the large circle equal 10. Then the ratio is:

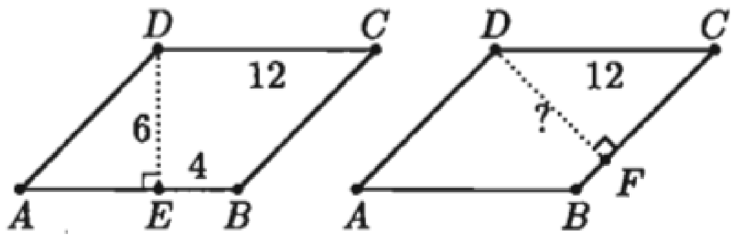
$$\frac{\begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{R+S}\end{pmatrix} - \begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{S}\end{pmatrix} + \begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{T}\end{pmatrix}}{\begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{T+U}\end{pmatrix} - \begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{T}\end{pmatrix} + \begin{pmatrix}\text{Area of}\\ \text{semicircle}\\ \text{S}\end{pmatrix}}$$

$$\frac{\frac{1}{2}\pi 5^2 - \frac{1}{2}\pi 2^2 + \frac{1}{2}\pi 3^2}{\frac{1}{2}\pi 5^2 - \frac{1}{2}\pi 3^2 + \frac{1}{2}\pi 2^2} = \frac{15\pi}{10\pi} = \frac{3}{2} \text{ or } 3:2.$$



1995 Q24

24. In parallelogram $ABCD$, $\overline{DE}$ is the altitude to the base $\overline{AB}$ and $\overline{DF}$ is the altitude to the base $\overline{BC}$. [**Note:** *Both pictures represent the same parallelogram.*] If $DC = 12$, $EB = 4$ and $DE = 6$, then $DF =$



(A) 6.4 (B) 7 (C) 7.2
(D) 8 (E) 10

24. (C) Since opposite sides of a parallelogram are equal, $AB = 12$. Then $AE = 12 - 4 = 8$. Using the Pythagorean Theorem gives $AD = \sqrt{8^2 + 6^2} = 10$, and then $BC = 10$ also. The area of a parallelogram is base $\times$ altitude. Using base $AB = 12$ and altitude $DE = 6$ gives an area of $12 \times 6 = 72$. Using base $BC = 10$ and altitude $\overline{DF}$ must also give an area of 72. Thus $DF = 72/10 = 7.2$.

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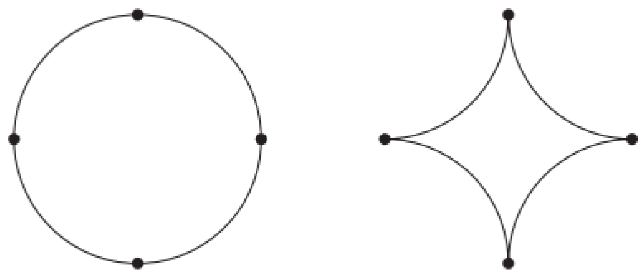
2012 Q24

24. A circle of radius 2 is cut into four congruent arcs. The four arcs are joined to form the star figure shown. What is the ratio of the area of the star figure to the area of the original circle?



(A) $\dfrac{4-\pi}{\pi}$ (B) $\dfrac{1}{\pi}$ (C) $\dfrac{\sqrt{2}}{\pi}$ (D) $\dfrac{\pi-1}{\pi}$ (E) $\dfrac{3}{\pi}$

24. **Answer (A):** Translate the star into the circle so that the points of the star coincide with the points on the circle. Construct four segments connecting the consecutive points of the circle and the star, creating a square concentric to the circle.

The area of the circle is $\pi(2)^2 = 4\pi$. The square is made up of four congruent right triangles with area $\frac{1}{2}(2 \times 2) = 2$, so the area of the square is $4 \times 2 = 8$. The area inside the circle but outside the square is $4\pi - 8$.

This is also the area inside the square but outside the star. So, the area of the star is $8 - (4\pi - 8) = 16 - 4\pi$. The ratio of the area of the star figure to the area of the original circle is $\frac{16-4\pi}{4\pi} = \frac{4-\pi}{\pi}$.