MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

33rd Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 14, 2017

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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Correspondence about the problems and solutions should be sent by email to:

amchq@maa.org

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MAA American Mathematics Competitions
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1. **Answer (A):** The values of the expressions are, in order, 10, 8, 9, 9, and 0.

2. **Answer (E):** Observe that 36 votes made up 30% of the total number of votes. Thus 12 votes made up 10% of the total number of votes and therefore there were 120 total votes.

3. **Answer (C):** Simplifying yields

$$\sqrt{16\sqrt{8\sqrt{4}}} = \sqrt{16\sqrt{16}} = \sqrt{64} = 8$$

4. **Answer (D):** The product may be estimated as

$$\left(3 \cdot 10^{-4}\right)\left(8 \cdot 10^{6}\right) = 24 \cdot 10^{2} = 2400.$$

   The exact value of the product is 2497.498.

5. **Answer (B):** The sum $1 + 2 + 3 + \cdots + 8 = 36$, so the desired quotient is

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{36} = 4 \cdot 5 \cdot 7 \cdot 8 = 1120.$$

6. **Answer (D):** Let the degree measures of the angles of the triangle be $3x$, $3x$, and $4x$. Then $3x + 3x + 4x = 10x = 180$, and $x = 18$. So the largest angle has degree measure $4x = 4 \cdot 18 = 72$.

OR

   The degree measure of the largest angle is $\frac{4}{3+3+4} = \frac{2}{5}$ of the sum of the degree measures of the angles in the triangle, so it is $\frac{2}{5} \cdot 180 = 72$ degrees.

7. **Answer (A):** Assume Z has the form *abcabc*. Then

$$Z = 1001 \cdot abc = 7 \cdot 11 \cdot 13 \cdot abc$$

   So 11 must be a factor of Z.

OR

   A positive integer is divisible by 11 if and only if the difference of the sums of the digits in the even and odd positions in the number is divisible by 11. For $Z = abcabc$ the sum of the digits in the even positions is equal to the sum of the digits in the odd positions, so the difference of the two sums is 0. Hence, 11 divides Z.

8. **Answer (D):** There are no two-digit even primes, so statements (1) and (2) cannot both be true. Also, no two-digit prime is divisible by 7, so statements (1) and (3) cannot both be true. Because there is only one false statement, it must be (1), so Isabella's house number is an even two-digit multiple of 7 that has a digit of 9. The number is even, so the 9 must be the tens digit. The only even multiple of 7 between 90 and 99 is 98, so the units digit is 8.

9. **Answer (D):** The total number of Marcy's marbles must be divisible by both 3 and 4 thus it must be a multiple of 12. If she has just 12 marbles, then 4 are blue and 3 are red, leaving just 5 other marbles, so she could not have 6 green marbles. If she has 24 marbles, then 8 are blue and 6 are red, leaving 10 other marbles, so she could have 6 green marbles and 4 yellow marbles. The smallest number of yellow marbles that Marcy would have is 4.

10. **Answer (C):** There are 10 possible equally likely outcomes:

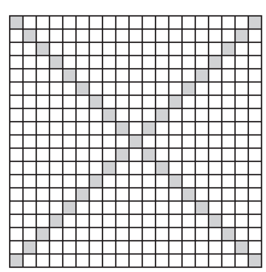
1, 2, 3	1, 2, 4	1, 2, 5	1, 3, 4	1, 3, 5
1, 4, 5	2, 3, 4	2, 3, 5	2, 4, 5	3, 4, 5

The three highlighted outcomes have 4 as the largest value selected. Hence the probability is $\frac{3}{10}$.

OR

There are 10 ways to select 3 cards without replacement from a box of 5 cards. If the largest value selected is 4, then the remaining two cards can be selected from the cards 1, 2, and 3 in 3 ways. So, the probability that 4 is the largest value selected is $\frac{3}{10}$.

11. **Answer (C):** If the total number of tiles in the two diagonals is 37, there are 19 tiles in each diagonal (with one tile appearing in both diagonals). The number of tiles on a diagonal is equal to the number of tiles on a side. Therefore, the square floor is covered by $19 \times 19 = 361$ square tiles.



12. **Answer (D):** The least common multiple of 4, 5, and 6 is 60. Numbers that leave a remainder of 1 when divided by 4, 5, and 6 are 1 more than a whole number multiple of 60. So the smallest positive number greater than 1 that leaves a remainder of 1 when divided by 4, 5, and 6 is 61.

13. **Answer (B):** Whenever a person wins a game, another person loses that game. So the total number of wins equals the total number of losses. Peter, Emma, and Kyler lost $2+3+3=8$ games altogether, and Peter and Emma won $4+3=7$ games in total. Therefore, Kyler won 1 game.

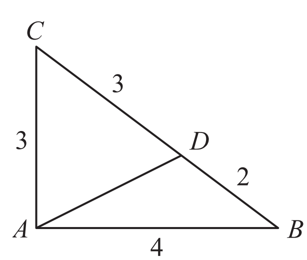
14. **Answer (C):** For simplicity, suppose that the assignment contained 100 problems. Chloe correctly solved 80% of the 50 problems she worked on alone, which was 40 problems. She had a total of 88 correct answers, so $88-40=48$ of the 50 problems that she and Zoe worked on together had correct answers. In addition Zoe correctly solved 90% of the 50 problems that she worked on alone, which was 45 problems. Her overall percentage of correct answers was $45+48=93$.

    OR

    Chloe's percentage of success on the half that she solved alone was 80, which is 8 points less than 88. So her percentage of success on the other half was 8 points above 88, or 96. Zoe's percentage of success was 90 on half of the problems and 96 on the other half, so her overall percentage was the average of 90 and 96, which is 93.

15. **Answer (D):** Starting at A, there are 4 choices for the M, each of which is followed by 3 choices for the C, each of which is followed by 2 choices for the 8. So all together, there are $4 \cdot 3 \cdot 2 = 24$ paths.

16. **Answer (D):** Because the perimeters of $\triangle ADC$ and $\triangle ADB$ are equal, $CD=3$ and $BD=2$.

    $\triangle ADC$ and $\triangle ADB$ have the same altitude from A, so the area of $\triangle ADC$ will be $3/5$ of the area of $\triangle ABC$, and $\triangle ADB$ will be $\frac{2}{5}$ of the area of $\triangle ABC$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$, so the area of $\triangle ADB$ is $\frac{2}{5} \cdot 6 = 12/5$.

17. **Answer (C):** After putting 9 coins in all but two of the chests, I could take 3 coins out of each chest to leave 6 coins in those chests. Doing this would allow me to fill the remaining 2 chests with 6 coins each, and have another 3 coins left over. So I must have removed 15 coins (3 from each of 5 chests). Thus I initially had put 9 coins into each of 5 chests, making for a total of 45 coins (and 7 chests).

OR

Let c be the number of chests. Then $9(c-2) = 6c+3$. Solving yields $c = 7$, so the number of coins is $6 \cdot 7 + 3 = 9(7-2) = 45$.

18. **Answer (B):** In right triangle BCD, $3^2 + 4^2 = 5^2$, so $BD = 5$. In $\triangle ABD$, $13^2 = 12^2 + 5^2$, so $\triangle ABD$ is a right triangle with right angle $\angle ABD$. The area of $\triangle ABD$ is $\frac{1}{2} \cdot 5 \cdot 12 = 30$. The area of $\triangle BCD$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. So the area of the quadrilateral is $30 - 6 = 24$.

19. **Answer (D):** Factoring yields $98! + 99! + 100! = 98!(1 + 99 + 100 \cdot 99) = 98!(100 + 100 \cdot 99) = 98!(100)(1+99) = 98! \cdot 100^2$. The exponent of 5 in 98! is $19 + 3 = 22$, one for each multiple of 5 and one more for each multiple of 25. Thus the exponent of 5 in the product is $22 + 4 = 26$ as $100^2 = 2^4 \cdot 5^4$.

20. **Answer (B):** There are 9000 integers between 1000 and 9999 inclusive. For an integer to be odd it must end in 1, 3, 5, 7, or 9. So there are 5 choices for the units digit. For a number to be between 1000 and 9999 the thousands digit must be nonzero and so there are now 8 choices for the thousands digit. For the hundreds digit there are 8 choices and for the tens digit there are 7 choices for a total number of $5 \cdot 8 \cdot 8 \cdot 7 = 2240$ choices. So the probability is $\frac{2240}{9000} = \frac{56}{225}$.

21. **Answer (A):** Since $a + b + c = 0$, then these three numbers cannot be all positive or all negative. The value of $\dfrac{X}{|X|} = 1$ for X positive and -1 for X negative.

Case I. When there are two positive numbers and one negative number,

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = 1,$$

and $\dfrac{abc}{|abc|} = -1$, so

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|} = 0.$$

Case II. When there are two negative numbers and one positive number,

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}=-1,$$

and $\dfrac{abc}{|abc|}=1$, so

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}=0.$$

Therefore the only possible value of $\dfrac{a}{|a|}+\dfrac{b}{|b|}+\dfrac{c}{|c|}+\dfrac{abc}{|abc|}$ is 0.

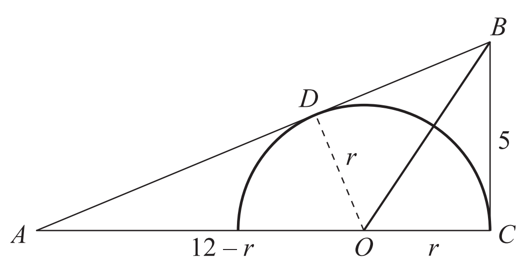
22. **Answer (D):** Let O be the center of the inscribed semicircle on $\overline{AC}$, and let D be the point at which $\overline{AB}$ is tangent to the semicircle. Because $\overline{OD}$ is a radius of the semicircle it is perpendicular to $\overline{AB}$, making $\overline{OD}$ an altitude of $\triangle AOB$. By the Pythagorean Theorem, $AB=13$. In the diagram, $\overline{OB}$ partitions $\triangle ABC$ so that

$$\text{Area}(\triangle ABC)=\text{Area}(\triangle BOC)+\text{Area}(\triangle AOB)$$

Since we know $\triangle ABC$ has area 30, we have

$$\begin{aligned}30 &= \text{Area}(\triangle BOC)+\text{Area}(\triangle AOB)\\ &= \frac{1}{2}(BC)r+\frac{1}{2}(AB)r=\frac{5}{2}r+\frac{13}{2}r=9r.\end{aligned}$$

Therefore $r=\frac{30}{9}=\frac{10}{3}$.



OR

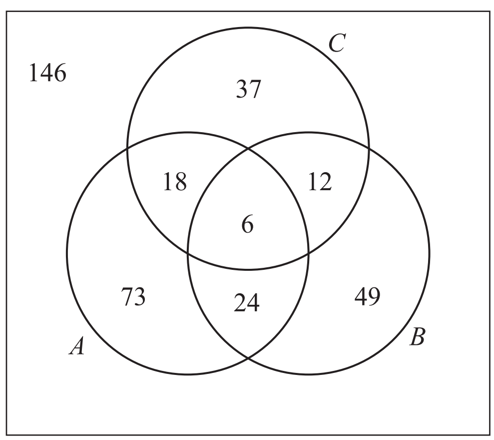
Because $\overline{OD}$ is a radius of the semicircle, it is perpendicular to $\overline{AB}$, making $\triangle ADO$ similar to $\triangle ACB$. Because $\overline{BC}$ and $\overline{BD}$ are both tangent to the semicircle, they are congruent. So $BD=5$ and $AD=8$. It follows that $\frac{r}{8}=\frac{5}{12}$ and so $r=\frac{40}{12}=\frac{10}{3}$.

23. **Answer (C):** Her time for each trip was 60 minutes. The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Her daily number of minutes for a mile form a sequence of four numbers where each number is 5 more than the previous number. Also, these numbers must each be a factor of 60 since the number of miles traveled must be an integer. The only such sequence from among the factors of 60 is 5, 10, 15, 20. So her rates in miles per minute for the four days were $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{20}$, and multiplying each by 60 minutes gives her distances in miles as 12, 6, 4, and 3, for a total distance of 25 miles.

24. **Answer (D):** During a 60-day cycle, there are 20 days that the first one calls, 15 days that the second one calls, and 12 days that the third one calls. The sum $20 + 15 + 12 = 47$ over-counts the number of days when more than one grandchild called. There were $60 \div 12 = 5$ days when the first and second called. There were $60 \div 15 = 4$ days when the first and third called. There were $60 \div 20 = 3$ days when the second and third called. Subtracting $5 + 4 + 3$ from 47 leaves 35 days. But the 60^{th} day was added in three times and subtracted out three times, so there were 36 days in which she received at least one phone call. Thus, in each 60-day cycle, there were $60 - 36 = 24$ days without a phone call. In a year, there are six full cycles. Additionally, she receives no phone call on the 361^{st} or 362^{nd} day. Therefore, the total number of days that she does not receive a phone call is $6 \cdot 24 + 2 = 146$ days.

OR

In the Venn diagram below, let A be the set of days in the year in which the first grandchild calls her, let B be the set of days in which the second calls her, and let C be the set of days in which the third calls her. Then the region common to the 3 sets represents the days on which all 3 call, which are every $3 \cdot 4 \cdot 5 = 60$ days, or 6 days in the year.

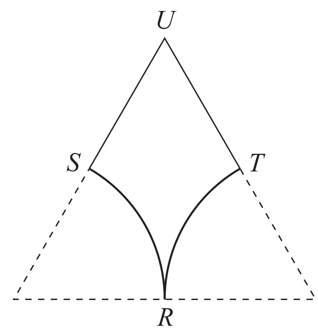


The region common to A and B represents the days on which the first and second grandchild call, which are every $3 \cdot 4 = 12$ days, or 30 days in the year. Since 6 of those days have already been counted, we label the region below the 6 with $30 - 6 = 24$. Similarly, the region common to A and C is 24, so the region to the left of the common region is $24 - 6 = 18$, and the region to the right is $18 - 6 = 12$.

Now the first grandchild calls on 121 days, of which 48 have already been counted. Thus the lower left region contains 73 days. Similarly, B contains 91 days, so the lower right region contains 49 days, and the top region has 37 days.

All the regions total 219 days, so the number of days without a call is $365 - 219 = 146$ days. Note that in leap years, the lower left region increases to 74, so the answer is still $366 - 220 = 146$ days.

25. **Answer (B):** The region shown is what remains when two one-sixth sectors of a circle of radius 2 are removed from an equilateral triangle with side length 4.



The area of an equaliateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. Thus the area of the region is $4\sqrt{3} - 2\left(\frac{1}{6} \cdot 4\pi\right) = 4\sqrt{3} - \frac{4\pi}{3}$.

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