MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

32nd Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 15, 2016

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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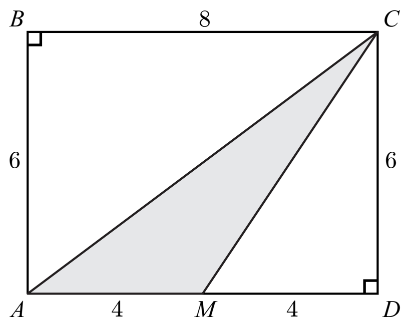
1. Answer (C):

   There are 60 minutes in 1 hour, so 11 hours plus 5 minutes is equal to $11 \cdot 60 + 5 = 665$ minutes.

2. Answer (A):

   The area of $\triangle ACD$ is $\frac{1}{2} \cdot 8 \cdot 6 = 24$. The area of $\triangle MCD$ is $\frac{1}{2} \cdot 4 \cdot 6 = 12$. So the area of $\triangle AMC$ is $24 - 12 = 12$.

OR



As seen in the diagram above, the altitude from C to the line of the base $\overline{AM}$ is $\overline{CD}$. Thus the area of the shaded $\triangle AMC$ is

$$\frac{1}{2} \cdot AM \cdot CD = \frac{1}{2} \cdot 4 \cdot 6 = 12.$$

3. Answer (A):

   The given scores 70, 80, and 90 are a total of 30 above the stated average. Thus the remaining score is 30 points below the average, and $70 - 30 = 40$.

OR

Let x be the missing score. Then the sum $70 + 80 + 90 + x = 70 \cdot 4 = 280$. So x must be 40.

4. Answer (B):

   As a boy it took Cheenu 3 hours and 30 minutes, which is 210 minutes, to go 15 miles. That is a rate of $210 \div 15 = 14$ minutes per mile. As an old man it takes him 4 hours, or 240 minutes, to travel 10 miles. That is a rate of $240 \div 10 = 24$ minutes per mile. It takes him $24 - 14 = 10$ minutes more to walk a mile as an old man.

5. Answer (E):

   The two-digit numbers that leave a remainder of 3 when divided by 10 are: 13, 23, 33, 43, 53, 63, 73, 83, 93. The two-digit numbers that leave a remainder of 1 when divided 9 are: 10, 19, 28, 37, 46, 55, 64, 73, 82, 91. Among these two sets, 73 is the only common number. When 73 is divided by 11 the remainder is 7.

6. Answer (B):

   The 19 name lengths are 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 7, 7. The tenth value, 4, is the median.

7. Answer (B):

   The numbers 1^{2016}, 3^{2018}, 5^{2020} have even exponents and hence are squares. The number 2^{2017} is not a perfect square because it is twice a square $2(2^{1008})^2$. SInce $4^{2019} = (2^2)^{2019} = 2^{4038}$, it is also a perfect square.

   OR

   A positive integer power of a square is again a square. This eliminates choices (A) and (D). An even power of any integer is a square. This eliminates choices (C) and (E) The only remaining choice is (B), and in fact, an odd power of a non-square cannot be a square.

8. Answer (C) :

   Evaluate the expression by grouping as follows:

   $(100-98)+(96-94)+\cdots+(8-6)+(4-2) = 2+2+\cdots+2+2 = 2\cdot 25 = 50.$

9. Answer (B):

   The prime factorization of 2016 is: $2016 = (2^5)(3^2)(7)$, so the distinct prime divisors of 2016 are 2, 3, and 7, and their sum is $2+3+7 = 12$.

10. Answer (D):

    Since $2*(5*x) = 1$, it follows that $6-(5*x) = 1$, and so $5*x = 5$. Applying the formula again, $15-x = 5$, and therefore $x = 10$.

11. Answer (B):

    Let $\underline{a}\,\underline{b}$ be the two digit number. Then $132 = (10a+b)+(10b+a) = 11(a+b)$.

    Thus $a+b = 12$. The possible numbers are: 39, 93, 48, 84, 57, 75, and 66. There are seven two-digit numbers that meet this criterion.

12. Answer (B):

    Converting the given fractions to the same denominator, we see that $\frac{9}{12}$ of the girls and $\frac{8}{12}$ of the boys went on the trip. So the ratio of the number of girls to the number of boys was $9:8$, and it follows that $\frac{9}{17}$ of the students on the trip were girls.

    OR

    The number of boys and girls must be a common multiple of 4 and 3, the denominators of the fractions given in the problem. Suppose there are 12 boys and 12 girls in Jefferson Middle School. Then 9 girls and 8 boys went on the trip, for a total of 17 students. The fraction of girls on the trip is 9/17.

13. Answer (D):

    There are $6 \cdot 5 = 30$ possible pairs of numbers. For a product to be 0, either the first factor or the second factor must be 0, so there are $1 \cdot 5 + 5 \cdot 1 = 10$ such products. The desired probability is $10/30 = 1/3$.

14. Answer (A):

    In driving 350 miles, Karl used $\frac{350}{35} = 10$ gallons of gas, so he had $14 - 10 = 4$ gallons left in his tank. After buying 8 more gallons, he had $4 + 8 = 12$ gallons. When he arrived at his destination, he had $\frac{14}{2} = 7$ gallons left, so he used an additional $12 - 7 = 5$ gallons. This let him drive an additional $5 \cdot 35 = 175$ miles, so he drove a total of $350 + 175 = 525$ miles.

    OR

    Karl used $14 + 8 - \frac{14}{2} = 15$ gallons of gas on his trip, so he drove $15 \cdot 35 = 525$ miles.

15. Answer (C):

    Factor, using a difference of two squares:

$$\begin{aligned} 13^4 - 11^4 &= \left(13^2 + 11^2\right)(13 + 11)(13 - 11) \\ &= 290 \cdot 24 \cdot 2 \\ &= 2 \cdot 145 \cdot 8 \cdot 3 \cdot 2 \\ &= 32 \cdot 435 \end{aligned}$$

    So the largest power of 2 that is a divisor of $13^4 - 11^4$ is 32.

16. Answer (D):

    Let N be the number of laps run by Annie when she passes Bonnie for the first time. The number of laps run by Bonnie is $N-1$. Then $\frac{N}{N-1} = 1.25 = \frac{5}{4}$. So $N = 5$.

    OR

    For each lap Bonnie completes, Annie runs $1^1/_4$ laps, thus gaining $^1/_4$ of a lap on Bonnie during that time. Annie will pass Bonnie when Bonnie has run 4 laps, at which point Annie will have run 5.

17. Answer (D):

    If there were no restrictions, then 10^4 passwords would be possible. Among these, 10 passwords begin with 9 1 1, and have 10 options for the fourth digit. Thus $10^4 - 10 = 9990$ passwords satisfy the condition.

18. Answer (C):

    Divide the 216 sprinters into 36 groups of 6. Run 36 races to eliminate 180 sprinters, leaving 36 winners. Divide the 36 winners into 6 groups of 6, run 6 races to eliminate 30 sprinters, leaving 6 winners. Finally run the last race to determine the champion. The number of races run is $36 + 6 + 1 = 43$.

    OR

    When all the races have been run, 215 sprinters will have been eliminated. Since 5 sprinters are eliminated in each race, there are $\frac{215}{5} = 43$ races needed to determine the champion.

19. Answer (E):

    The average of the 25 even integers is $10000/25 = 400$. So 12 consecutive even integers will be larger than 400 and 12 consecutive even integers will be smaller than 400. The sum $376 + 378 + \cdots + 398 + 400 + 402 + \cdots + 422 + 424 = 10000$. The largest of these numbers is 424.

    OR

    The average of the 25 even integers is $\frac{10000}{25} = 400$. Since 12 of the consecutive even integers are larger than 400, the largest is $400 + 12 \cdot 2 = 424$.

20. Answer (A):

If $b = 1$, then $a = 12$ and $c = 15$, and the least common multiple of a and c is 60. If $b > 1$, then any prime factor of b must also be a factor of both 12 and 15, and thus the only possible value is $b = 3$. In this case, a must be a multiple of 4 and a divisor of 12, so $a = 4$ or $a = 12$. Similarly, c must be a multiple of 5 and a divisor of 15, so $c = 5$ or $c = 15$. It follows that the least common multiple of a and c must be a multiple of 20. When $a = 4$, $b = 3$, and $c = 5$, the least common multiple of a and c is exactly 20.

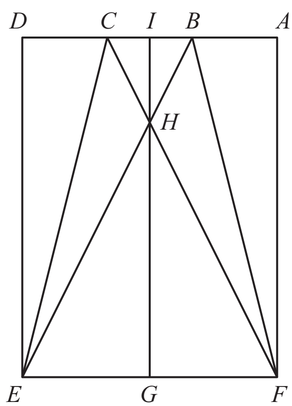
21. Answer (B):

Consider drawing all five chips and listing the 10 possible outcomes: RRRGG, RRGRG, RGRRG, GRRRG, GGRRR, GRGRR, RGGRR, GRRGR, RGRGR, RRGGR.

All 10 of these outcomes are equally likely. The outcomes that end in G correspond to the outcomes where the 3 reds are drawn and the outcomes that end in R correspond to the outcomes where the 2 greens are drawn. The probability that the 3 reds are drawn is $\frac{4}{10} = \frac{2}{5}$.

22. Answer (C):

The area of $\triangle BCE$ is $\frac{1}{2}(1)(4) = 2$. Triangles $\triangle CBH$ and $\triangle EFH$ are similar. Since $CB = \frac{1}{3}EF$, it follows that $IH = \frac{1}{3}GH = \frac{1}{4}IG = 1$. The area of $\triangle CBH$ is $\frac{1}{2}$, so the area of $\triangle ECH$ is $2 - \frac{1}{2} = \frac{3}{2}$. Thus the batwing's area is 3.

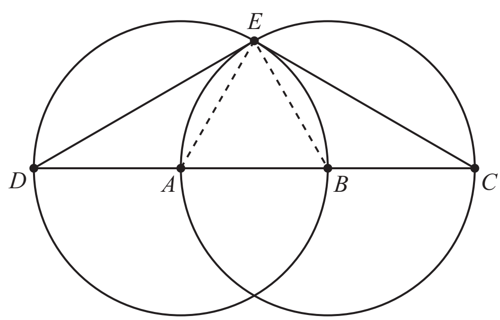


23. Answer (C):

We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measure of $\angle AEB$ is 60°. Since $\overline{DB}$ is a diameter of circle A and $\overline{AC}$ is a diameter of circle B, it follows that $\angle DEB$ and $\angle AEC$ are both right angles. Therefore the degree measure of $\angle DEC$ is $90^\circ + 90^\circ - 60^\circ = 120^\circ$.

OR

We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measures of $\angle AEB$ and $\angle EAB$ are both 60°. Then the measure of $\angle DAE$ is 120°, and since $\triangle DAE$ is isosceles, the measure of $\angle DEA$ is 30°. Similarly, the measure of $\angle BEC$ is also 30°. Therefore the degree measure of $\angle DEC$ is $30^\circ + 60^\circ + 30^\circ = 120^\circ$.



24. Answer (A):

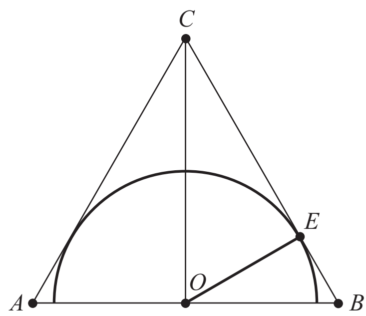
Since QRS is divisible by 5, we know that $S = 5$. Since PQR is divisible by 4, we know that QR is 12, 32, or 24. So RST will be either $25T$ or $45T$ and divisible by 3. Using the available digits, 453 is the only number that is divisible by 3. So $T = 3$, $R = 4$, and $P = 1$.

25. Answer (B):

Let O be the midpoint of base $\overline{AB}$ of $\triangle ABC$ and the center of the semicircle. Triangle $\triangle OBC$ is a right triangle with $OB = 8$ and $OC = 15$, and so, by the Pythagorean Theorem, $BC = 17$. Let E be the point where the semicircle intersects $\overline{BC}$, so radius $\overline{OE}$ is perpendicular to $\overline{BC}$. Then $\triangle OEB$ and $\triangle COB$ are similar, and therefore, $OE : CO = OB : CB$. Hence, $\frac{OE}{15} = \frac{8}{17}$ and so $OE = \frac{120}{17}$.

OR

Let O be the center of the semicircle, which is also the midpoint of base $\overline{AB}$. Since $OB = 8$ and $OC = 15$, then by the Pythagorean Theorem $BC = 17$. Let E be the point where the semicircle intersects $\overline{BC}$, so radius $\overline{OE}$ is perpendicular to $\overline{BC}$. Since the area of $\triangle OBC$ is $\frac{1}{2}(BC)(OE) = \frac{1}{2}(OB)(OC)$, then $\frac{1}{2}(17)(OE) = \frac{1}{2}(8)(15)$ and so $OE = 120/17$.



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