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# Solutions Pamphlet

American Mathematics Competitions

30th Annual

# AMC 8

**American Mathematics Contest 8**

**Tuesday, November 18, 2014**

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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1. **Answer (A):** Harry's answer is $H = 8 - (2 + 5) = 8 - 7 = 1$. Terry's answer is $T = 8 - 2 + 5 = 6 + 5 = 11$. The difference $H - T$ is $1 - 11 = -10$.

2. **Answer (E):** To use the largest number of coins, Paul would use only 5-cent coins. Because $35 = 5 \cdot 7$, the largest number of coins Paul can use is 7. To use the smallest number of coins, Paul would use a 25-cent coin and a 10-cent coin, for a total of 2 coins. The difference between the largest and the smallest number of coins he can use is $7 - 2 = 5$.

3. **Answer (B):** The number of pages in the book is

$$3 \cdot 36 + 3 \cdot 44 + 10 = 3(36 + 44) + 10 = 3 \cdot 80 + 10 = 250.$$

4. **Answer (E):** The sum of two odd primes is an even number. Since the sum 85 is odd, one of the primes must be 2, which is the only even prime. The two primes are 2 and 83, so the product is $2 \cdot 83 = 166$.

5. **Answer (C):** For \$20, Margie can buy $\frac{20}{4} = 5$ gallons of gas. She can drive 32 miles on each gallon, for a total of $32 \cdot 5 = 160$ miles.

6. **Answer (D):** The areas of the six rectangles are 2, 8, 18, 32, 50, and 72. Adding yields 182.

**OR**

The sum of areas is

$$2 \cdot 1 + 2 \cdot 4 + 2 \cdot 9 + 2 \cdot 16 + 2 \cdot 25 + 2 \cdot 36 = 2(1 + 4 + 9 + 16 + 25 + 36) = 2 \cdot 91 = 182.$$

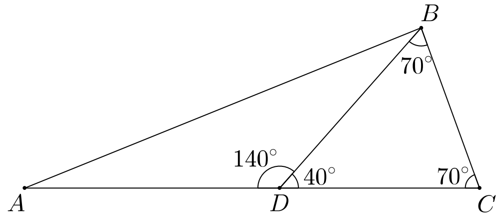
7. **Answer (B):** If there were an equal number of girls and boys, there would be 14 of each. By increasing the number of girls by 2 and decreasing the number of boys by 2, we see that there are 16 girls and 12 boys for a ratio of $16 : 12$ or $4 : 3$.

**OR**

If there were 4 fewer girls, then the class would be half boys and half girls. Remove 4 girls from the 28, and the other 24 students are evenly split into 12 boys and 12 girls. Add back the 4 girls to get 16 girls and 12 boys for a $16 : 12$ ratio, which simplifies to $4 : 3$.

8. **Answer (D):** The multiples of 11 between 102 and 192 are 110, 121, 132, 143, 154, 165, 176, and 187. Only 132 satisfies the condition, so $A = 3$.

9. **Answer (D):** Triangle $BCD$ is isosceles, so $\angle BCD = \angle CBD = 70°$ and $\angle BDC = 180° - 2 \cdot 70° = 40°$. Hence $\angle ADB = 180° - 40° = 140°$.

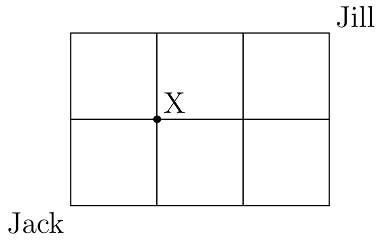



10. **Answer (A):** The seventh AMC 8 was given in 1991. So Samantha was born in $1991 - 12 = 1979$.

**OR**

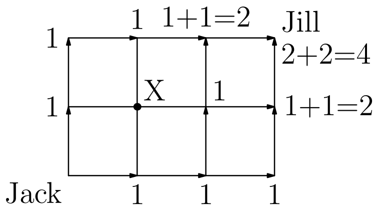
Because the seventh AMC 8 was given when Samantha was 12, the first was 6 years earlier and she was 6 that first year in 1985. She was born 6 years earlier, in 1979.

11. **Answer (A):** Let E represent traveling a block east and N represent traveling a block north. To avoid the dangerous intersection the first two blocks must be EE or NN. So there 4 possible paths: EEENN, EENEN, EENNE, and NNEEE.




**OR**

In the following diagram, the numbers indicate the number of ways to get to each of the intersections. In each case, the number of ways to get to any particular intersection is the sum of the numbers of ways to get to any of the intersections leading directly to it. Thus, there are four paths to Jill's house, avoiding the dangerous intersection.




12. **Answer (B):** Call the celebrities $L$, $M$, and $N$. There are six possible orderings: $LMN$, $LNM$, $MLN$, $MNL$, $NLM$, and $NML$. Only one of these identifies all three correctly. Therefore the probability is $\frac{1}{6}$.

13. **Answer (D):** If $n^2 + m^2$ is even, then $n^2$ and $m^2$ are either both even or both odd, which means $n$ and $m$ are either both even or both odd. If $n$ and $m$ are both even, their sum is even. If $n$ and $m$ are both odd, their sum is even. Because $n + m$ is never odd, (D) is the impossible choice.

14. **Answer (B):** The area of rectangle $ABCD$ is $5 \cdot 6 = 30$. The area of triangle $DCE$ is also 30, which is half of the product $CD \cdot CE$, so that product is 60. Because $CD = AB = 5$, $CE$ must equal $\frac{60}{5} = 12$, and by the Pythagorean Theorem, $DE = \sqrt{CD^2 + CE^2} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$.

15. **Answer (C):** Angle $AOE$ is $\frac{4}{12}$ of $360°$ or $120°$ degrees, while $\angle GOI$ is $\frac{2}{12}$ of $360°$ or $60°$. Both triangles are isosceles, so the equal base angles are $\frac{60°}{2}$ and $\frac{120°}{2}$ respectively. The sum of angles $x$ and $y$ then is $(60° + 120°)/2 = 90°$.

16. **Answer (B):** Each team plays 4 non-conference games for a total of 32 games against non-conference opponents. Each team plays 7 conference games at home for a total of 56 games within the conference. The total number of games is $32 + 56 = 88$.

17. **Answer (B):** To walk 1 mile at 3 miles per hour requires $\frac{1}{3}$ of an hour, or 20 minutes. This is the amount of time George allows himself to get to school. To walk $\frac{1}{2}$ mile at 2 miles per hour requires $\frac{\frac{1}{2}}{2} = \frac{1}{4}$ of an hour, or 15 minutes, so George has only 5 minutes to cover the remaining $\frac{1}{2}$ mile. Because 5 minutes is $\frac{5}{60} = \frac{1}{12}$ of an hour, George needs to run at a speed of $\frac{1/2}{1/12} = 6$ miles per hour.

18. **Answer (D):** The 16 equally likely outcomes may be grouped as follows:

4 boys: BBBB

3 boys, 1 girl: BBBG, BBGB, BGBB, GBBB

2 boys, 2 girls: BBGG, BGBG, BGGB, GBBG, GBGB, GGBB

1 boy, 3 girls: BGGG, GBGG, GGBG, GGGB

4 girls: GGGG

There are 8 equally likely outcomes that produce 3 of one gender and 1 of the other gender, so that result is most likely.

19. **Answer (A):** The amount of white surface area is smallest when you place one white cube in the interior of the larger cube. Place each of the other 5 white cubes at the center of a face so that 1 white face and 8 red faces are visible on that face. The total surface area of the larger cube is $6 \cdot 3^2 = 54$ square inches, so the fraction of the surface area that is white is $\frac{5}{54}$.

20. **Answer (B):** The areas of the quarter-circles are $\frac{\pi}{4}$, $\pi$ and $\frac{9\pi}{4}$. Their total area is $\frac{7\pi}{2}$. Using $\frac{22}{7}$ as an approximation of $\pi$, this is $\frac{7}{2} \cdot \frac{22}{7} = 11$, leaving $15 - 11 = 4$ for the desired area. (Using 3.14 for $\pi$ yields 4.01.)

21. **Answer (A):** For $\underline{7}\,\underline{4}\,\underline{A}\,\underline{5}\,\underline{2}\,\underline{B}\,\underline{1}$ to be a multiple of 3, the sum $7+4+A+5+2+B+1 = 19+A+B$ must be a multiple of 3. Therefore $A+B$ is 1 less than a multiple of 3. The sum $3+2+6+A+B+4+C = 15+A+B+C$ must be a multiple of 3, so $A+B+C$ must be a multiple of 3. Since $A+B$ is 1 less than a multiple of 3, $C$ must be 1 more than a multiple of 3. Only choice (A) meets this requirement.

22. **Answer (E):** Test ten consecutive numbers with unit's digits 0 through 9. For 10 through 19 we find that adding the product of the digits and the sum of the digits yields the sequence 1, 3, 5, 7, 9, 11, 13, 15, 17, and 19. In this sequence only 19 meets the desired condition. It is easy to verify that 29, 39, . . ., 99 also meet the desired condition.

**OR**

With $a$ as the tens digit and $b$ as the units digit, the number is $10a + b$. So $a + b + ab = 10a + b, a + ab = 10a, ab = 9a$ and $b = 9$.

23. **Answer (A):** The sum of any two of the girls' uniform numbers must be no greater than 31. The only possible sums of two 2-digit primes that are no greater than 31 are

$$11 + 13 = 24, \quad 11 + 17 = 28, \quad 11 + 19 = 30, \quad \text{and} \quad 13 + 17 = 30.$$

Therefore the required dates are 24, 28, and 30. Caitlin's uniform number must appear in the two smallest sums, $11 + 13 = 24$ and $11 + 17 = 28$. So Caitlin's uniform number is 11. The other two girl's uniform numbers are 13 for Ashley and 17 for Bethany.

24. **Answer (C):** Suppose the numbers of cans purchased by the 100 customers are listed in increasing order. The median is the average of the 50th and 51st numbers in the ordered list. To maximize the median, minimize the first 49 numbers by taking them all to be 1. If the 50th number is 4, then the sum of all 100 numbers would at least $49 + 51 \cdot 4 = 253$, which is too large. If instead the 50th number is 3 and the following numbers all equal 4, then the sum of the 100 numbers is $49 + 3 + 50 \cdot 4 = 252$ and the median is $(3 + 4) \div 2 = 3.5$.

**OR**

To maximize the median, the largest 50 should be the same and as large as possible. The lower 49 should be as small as possible. The median of the list will be the average of the 50th and 51st numbers. If every customer has 1 can of soda, there are 152 left to distribute. Giving the upper 50 three more each gives the top 50 four cans each (200 total) and the lower 50 one each (50 total). There are 2 cans left. Giving the 50th person the extra 2 means the 50th has 3 cans, and the 51st has 4 cans for a median of $(3 + 4) \div 2 = 3.5$.

25. **Answer (B):** Each semicircle moves Robert 40 feet ahead, so he would have to ride $5280 \div 40 = 132$ semicircles to cover 1 mile. Riding 132 semicircles is equal to the distance of 66 full circles. Each circle has a circumference of $40\pi$, so Robert rides $66 \cdot 40\pi$ feet. Converting to miles, that is $\frac{66 \cdot 40\pi}{5280} = \frac{\pi}{2}$ miles. Since he is riding at 5 miles per hour, it will take him $\frac{\pi}{2} \div 5 = \frac{\pi}{10}$ hours.

**OR**

Each semi-circular path is $\frac{\pi}{2}$ times as long as the straight path. Since the straight path would take $\frac{1}{5}$ hour to ride, the curved path will take $\frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10}$ hours to ride.

The problems and solutions in this contest were proposed by Steve Blasberg, Tom Butts, John Cocharo, Barb Currier, Steve Dunbar, Josh Frost, Joe Kennedy, Norb Kuenzi, Jeganathan Srikandarajah, and Dave Wells.

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