The Mathematical Association of America
American Mathematics Competitions

25th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 17, 2009

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the competition rules.*

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1. **Answer (E):** Work backwards. Bridget had 7 apples before she gave Cassie 3 apples. These 7 apples were half of Bridget's 14 original apples.

OR

Let $B =$ Bridget's original number of apples.

$$\frac{B}{2} - 3 = 4$$
$$\frac{B}{2} = 7$$
$$B = 14$$

So Bridget originally had 14 apples.

2. **Answer (D):** Let $s =$ number of sedans. Set up a proportion: $\frac{4}{7} = \frac{28}{s} = \frac{4(7)}{7(7)} = \frac{28}{49}$. So the dealership expects to sell 49 sedans.

OR

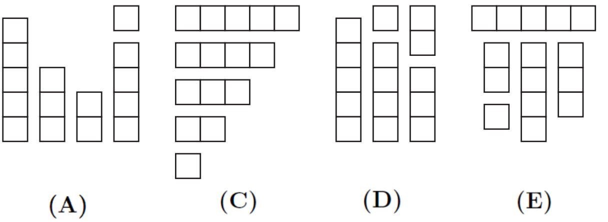
Because selling 4 sports cars corresponds to selling 7 sedans, 28 sports cars $=$ 7(4 sports cars) corresponds to 7(7 sedans) $=$ 49 sedans.

3. **Answer (C):** Suzanna rides at a constant rate of five minutes per mile. In 30 minutes there are six 5-minute intervals, so she travels six miles.

OR

Suzanna rides 3 miles in 15 minutes, so she will ride 6 miles in 30 minutes.

4. **Answer (B):** Figure B does not contain any 5-square-long piece. One solution is given for each of the other four figures. There are other solutions.



5. **Answer (D):** Make the list:

Position	1	2	3	4	5	6	7	8
Number	1	2	3	$1+2+3 = 6$	$2+3+6 = 11$	$3+6+11 = 20$	$6+11+20 = 37$	$11+20+37 = \mathbf{68}$

So the eighth number in the sequence is 68.

6. **Answer (A):** Together the hoses supply 10 gallons per minute to the pool. The pool holds 24,000 gallons, so it will take a total of $\dfrac{24{,}000 \text{ gallons}}{10 \text{ gallons/minute}} =$ 2400 minutes. Because 2400 minutes equals 40 hours, it takes 40 hours to fill Steve's pool.

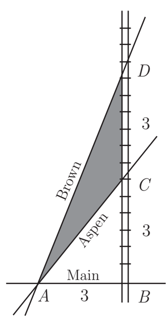
OR

The hoses supply (10 gallons/minute)(60 minutes/hour) = 600 gallons/hour. So it will take $\dfrac{24{,}000 \text{ gallons}}{600 \text{ gallons/hour}} = 40$ hours to fill Steve's pool.

7. **Answer (C):** The area of $\triangle ABC$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$ square miles. The area of $\triangle ABD = \frac{1}{2}(3)(6) = 9$ square miles. The shaded area is the area of $\triangle ABD$ minus the area of $\triangle ABC$, which is $9 - \frac{9}{2} = \frac{9}{2} = 4.5$ square miles.

OR

The base $\overline{CD}$ of $\triangle ACD$ is 3 miles. The altitude $\overline{AB}$ of $\triangle ACD$ is 3 miles. The area of $\triangle ACD$ is $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} = 4.5$ squares miles.



8. **Answer (B):** A rectangle with length L and width W has area LW. The new rectangle has area $(1.1)L \times (0.9)W = 0.99LW$. The new area $0.99LW$ is 99% of the old area.

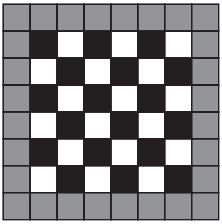
9. **Answer (B):** One side of the triangle and one side of the octagon will each touch one other polygon. Two sides of the other polygons will touch other polygons. Make a table and add the appropriate number of sides.

Number of sides	3	4	5	6	7	8
Number of sides that touch other polygons	1	2	2	2	2	1
Number of sides that don't	2	2	3	4	5	7

The resulting polygon has $2 + 2 + 3 + 4 + 5 + 7 = 23$ sides.

10. **Answer (D):** The checkerboard has 64 unit squares. There are $2\cdot 8+2\cdot 6=28$ unit squares on the outer edge, and $64-28=36$ unit squares in the interior. Therefore the probability of choosing a unit square that does not touch the outer edge is $\frac{36}{64}=\frac{18}{32}=\frac{9}{16}$.

OR



There are $(8-2)^2=36$ unit squares in the interior. Therefore, the probability of choosing a unit square that is does not touch the outer edge is $\frac{36}{64}=\frac{18}{32}=\frac{9}{16}$.

11. **Answer (D):** The number of sixth graders who bought a pencil is 195 divided by the cost of a pencil. Similarly the number of seventh graders who bought a pencil is 143 divided by the cost of a pencil. That means both 195 and 143 are multiples of the price of the pencil. Factor $195=1\cdot 3\cdot 5\cdot 13$ and $143=1\cdot 11\cdot 13$. The only common divisors are 1 and 13. If a pencil cost 1 cent, then 195 sixth graders bought a pencil. However, there are only 30 sixth graders, so a pencil must cost 13 cents. Using that fact, $\frac{195}{13}-\frac{143}{13}=15-11=4$ more sixth graders than seventh graders bought pencils.

12. **Answer (D):** Make a table.

	1	3	5
2	$1+2=3$	$3+2=5$	$5+2=7$
4	$1+4=5$	$3+4=7$	$5+4=9$
6	$1+6=7$	$3+6=9$	$5+6=11$

The table shows that seven of the nine equally likely events have prime numbers for their outcomes. So the probability of a prime outcome is $\frac{7}{9}$.

13. **Answer (B):** There are 6 three-digit numbers possible using the digits 1, 3 and 5 once each: 135, 153, 315, 351, 513 and 531. Because the numbers divisible by 5 end in 0 or 5, only 135 and 315 are divisible by 5. The probability that the three-digit number is divisible by 5 is $\frac{2}{6} = \frac{1}{3}$.

OR

The number is equally likely to end in 1, 3 or 5. The number is divisible by 5 only if it ends in 5, so the probability is $\frac{1}{3}$.

14. **Answer (B):** Find the time traveling to Temple by dividing the distance, 50 miles, by the rate, 60 miles per hour: $\frac{50}{60} = \frac{5}{6}$ hours. Find the time returning by dividing the distance, 50 miles, by the rate, 40 miles per hour: $\frac{50}{40} = \frac{5}{4}$ hours. Find the average speed for the round trip by dividing the total distance, $2 \cdot 50 = 100$ miles, by the total time, $\frac{5}{6} + \frac{5}{4} = \frac{10}{12} + \frac{15}{12} = \frac{25}{12}$ hours. The average speed is $\dfrac{100}{\frac{25}{12}} = 100(\frac{12}{25}) = 48$ miles per hour.

NOTE: The harmonic mean h of 2 numbers a and b is found using the formula $h = \frac{2ab}{a+b}$. The harmonic mean is the average rate if the same distance is traveled at two different rates.

If $a = 60$ and $b = 40$, then $h = \frac{2 \cdot 60 \cdot 40}{60+40} = \frac{4800}{100} = 48$ miles per hour.

15. **Answer (D):** Jordan has 5 squares of chocolate, which is $2\frac{1}{2}$ times the amount the recipe calls for. She has $2 \div \frac{1}{4} = 8$ times the amount of sugar and $\frac{7}{4} = 1\frac{3}{4}$ times the amount of milk necessary to make the recipe. So the amount of milk limits the number of servings. Jordan cannot make more than $5\left(1\frac{3}{4}\right) = 5\left(\frac{7}{4}\right) = \frac{35}{4} = 8\frac{3}{4}$ servings of hot chocolate.

16. **Answer (D):** The possible ways of expressing 24 as a product of 3 digits are $(1 \cdot 3 \cdot 8)$, $(1 \cdot 4 \cdot 6)$, $(2 \cdot 3 \cdot 4)$ and $(2 \cdot 2 \cdot 6)$. From the first product, the six integers 138, 183, 318, 381, 813 and 831 can be formed. Similarly, six integers can be formed from each of the products $(1 \cdot 4 \cdot 6)$ and $(2 \cdot 3 \cdot 4)$. From the product $(2 \cdot 2 \cdot 6)$, the three integers 226, 262 and 622 can be formed. The total number of integers whose digits have a product of 24 is $6 + 6 + 6 + 3 = 21$.

17. **Answer (B):** Factor 360 into $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. First increase the number of each factor as little as possible to form a square: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5)(2 \cdot 5) = (360)(10)$, so x is 10. Then increase the number of each factor as little as possible to form a cube: $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5)(3 \cdot 5 \cdot 5) = (360)(75)$, so y is 75. The sum of x and y is $10 + 75 = 85$.

18. **Answer (C):** To maintain the pattern, white squares will always occupy the corners, and every edge of the square pattern will have an odd number of tiles. Create a table, starting with a white square in the corner of the pattern, and increase the sides by 2 tiles.

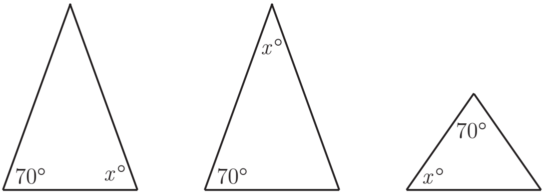
Floor area	# of white squares	Pattern
1×1	1	1^2
3×3	4	2^2
5×5	9	3^2
7×7	16	4^2
9×9	25	5^2

Following the pattern, an 11×11 area has 36 squares, a 13×13 area has 49, and a 15×15 has 64.

OR

There will be 8 rows that each contain 8 white tiles, so the total is $8(8) = 64$.

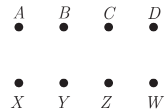
19. **Answer (D):** The two angles measuring 70° and x°, in an isosceles triangle, could be positioned in three ways, as shown.



If 70° and x° are the degree measures of the congruent angles, then $x = 70$. If x is the degree measure of the vertex, then x is $180 - 70 - 70 = 40$. If x is the degree measure of one of the base angles, but not 70, then x is $\frac{1}{2}(180 - 70) = 55$.

The possible values of x are 70, 40 and 55. The sum of these values is $70 + 40 + 55 = 165$.

20. **Answer (D):** With the points labeled as shown, one set of non-congruent triangles is AXY, AXZ, AXW, AYZ, AYW, AZW, BXZ and BXW.



Every other possible triangle is congruent to one of the 8 listed triangles.

CHALLENGE: Find the 48 distinct triangles possible and group them into sets of congruent triangles.

21. **Answer (D):** There are 40 rows, so the sum of the 40 row sums is $40A$. This number is also the sum of all of the numbers in the array because each number in the array is added to obtain one of the row sums. Similarly, there are 75 columns, so the sum of the 75 column sums is $75B$, and this, too, is the sum of all of the numbers in the array. So $40A = 75B$, and $\frac{A}{B} = \frac{75}{40} = \frac{15}{8}$.

22. **Answer (D):** There are 8 one-digit positive integers, excluding 1. There are $8 \cdot 9 = 72$ two-digit integers that do not contain the digit 1. There are $8 \cdot 9 \cdot 9 = 648$ three-digit integers that do not contain the digit 1. There are $8 + 72 + 648 = 728$ integers between 1 and 1000 that do not contain the digit 1.

OR

Think of each number between 1 and 1000 as a three-digit number. For example, think of 2 as 002 and 27 as 027. There are $9^3 = 729$ three-digit numbers that do not use the digit 1. Because 000 does not represent a whole number between 1 and 1000, the total is 728.

23. **Answer (B):** Mrs. Wonderful gave $400 - 6 = 394$ jelly beans to the class. Make a table, starting with a small, reasonable number of girls and boys.

Girls	Boys	Number of jelly beans
9	11	$(9 \times 9) + (11 \times 11) = 202$
10	12	$(10 \times 10) + (12 \times 12) = 244$
11	13	$(11 \times 11) + (13 \times 13) = 290$
12	14	$(12 \times 12) + (14 \times 14) = 340$
13	15	$(13 \times 13) + (15 \times 15) = 394$

The number of students is $13 + 15 = 28$.

24. **Answer (E):** Because $A + B = A$, $B = 0$. Subtracting 10 from A and adding 10 to B, $10 - A = A$, so A must be 5, and AB is 50. Because $50 - C5 = 5$, $C = 4$ and $D = 5 + 4 = 9$.

25. **Answer (E):** Looking from either end, the visible area totals $\frac{1}{2}$ square foot because piece A measures $\frac{1}{2} \times 1 = \frac{1}{2}$ ft^2, and the other pieces decrease in height from that piece. The two side views each show four blocks that can stack to a unit cube. So the area as seen from each side is 1 ft^2. Finally, the top and bottom views each show four unit squares. So the top and bottom view each contribute 4 ft^2 to the area. Summing, the total surface area is

$$\frac{1}{2} + \frac{1}{2} + 1 + 1 + 4 + 4 = 11 \text{ square feet.}$$

CHALLENGE: Suppose the cuts are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. Does this change the solution?