The Mathematical Association of America
American Mathematics Competitions

24th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 18, 2008

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the competition rules.*

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1. **Answer (B):** Susan spent $2 \times 12 = \$24$ on rides, so she had $50 - 12 - 24 = \$14$ to spend.

2. **Answer (A):** Because the key to the code starts with zero, all the letters represent numbers that are one less than their position. Using the key, C is $9 - 1 = 8$, and similarly L is 6, U is 7, and E is 1.

BEST	OF	LUCK
0 1 2 3	4 5	6 7 8 9

   CLUE = 8671

3. **Answer (A):** A week before the 13^{th} is the 6^{th}, which is the first Friday of the month. Counting back from that, the 5^{th} is a Thursday, the 4^{th} is a Wednesday, the 3^{rd} is a Tuesday, the 2^{nd} is a Monday, and the 1^{st} is a Sunday.

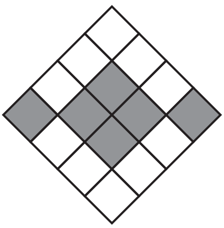
   OR

   Counting forward by sevens, February 1 occurs on the same day of the week as February 8 and February 15. Because February 13 is a Friday, February 15 is a Sunday, and so is February 1.

4. **Answer (C):** The area of the outer triangle with the inner triangle removed is $16 - 1 = 15$, the total area of the three congruent trapezoids. Each trapezoid has area $\frac{15}{3} = 5$.

5. **Answer (E):** Barney rides $1661 - 1441 = 220$ miles in 10 hours, so his average speed is $\frac{220}{10} = 22$ miles per hour.

6. **Answer (D):** After subdividing the central gray square as shown, 6 of the 16 congruent squares are gray and 10 are white. Therefore, the ratio of the area of the gray squares to the area of the white squares is $6 : 10$ or $3 : 5$.



7. **Answer (E):** Note that $\frac{M}{45} = \frac{3}{5} = \frac{3\cdot 9}{5\cdot 9} = \frac{27}{45}$, so $M = 27$. Similarly, $\frac{60}{N} = \frac{3}{5} = \frac{3\cdot 20}{5\cdot 20} = \frac{60}{100}$, so $N = 100$. The sum $M + N = 27 + 100 = 127$.

OR

Note that $\frac{M}{45} = \frac{3}{5}$, so $M = \frac{3}{5} \cdot 45 = 27$. Also $\frac{60}{N} = \frac{3}{5}$, so $\frac{N}{60} = \frac{5}{3}$, and $N = \frac{5}{3} \cdot 60 = 100$. The sum $M + N = 27 + 100 = 127$.

8. **Answer (D):** The sales in the 4 months were \$100, \$60, \$40 and \$120. The average sales were $\dfrac{100 + 60 + 40 + 120}{4} = \dfrac{320}{4} = \$80$.

OR

In terms of the \$20 intervals, the sales were 5, 3, 2 and 6 on the chart. Their sum is $5 + 3 + 2 + 6 = 16$ and the average is $\frac{16}{4} = 4$. The average sales were $4 \cdot \$20 = \$80$.

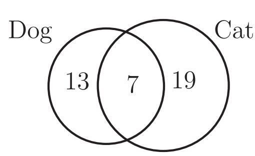
9. **Answer (D):** At the end of the first year, Tammy's investment was 85% of the original amount, or \$85. At the end of the second year, she had 120% of her first year's final amount, or 120% of $\$85 = 1.2(\$85) = \$102$. Over the two-year period, Tammy's investment changed from \$100 to \$102, so she gained 2%.

10. **Answer (D):** The sum of the ages of the 6 people in Room A is $6 \times 40 = 240$. The sum of the ages of the 4 people in Room B is $4 \times 25 = 100$. The sum of the ages of the 10 people in the combined group is $100 + 240 = 340$, so the average age of all the people is $\frac{340}{10} = 34$.

11. **Answer (A):** The number of cat owners plus the number of dog owners is $20 + 26 = 46$. Because there are only 39 students in the class, there are $46 - 39 = 7$ students who have both.

OR

Because each student has at least a cat or a dog, there are $39 - 20 = 19$ students with a cat but no dog, and $39 - 26 = 13$ students with a dog but no cat. So there are $39 - 13 - 19 = 7$ students with both a cat and a dog.



12. **Answer (C):** The table gives the height of each bounce.

Bounce	1	2	3	4	5
Height in Meters	2	$\frac{2}{3}\cdot 2 = \frac{4}{3}$	$\frac{2}{3}\cdot\frac{4}{3} = \frac{8}{9}$	$\frac{2}{3}\cdot\frac{8}{9} = \frac{16}{27}$	$\frac{2}{3}\cdot\frac{16}{27} = \frac{32}{81}$

Because $\frac{16}{27} > \frac{16}{32} = \frac{1}{2}$ and $\frac{32}{81} < \frac{32}{64} = \frac{1}{2}$, the ball first rises to less than 0.5 meters on the fifth bounce.
Note: Because all the fractions have odd denominators, it is easier to double the numerators than to halve the denominators. So compare $\frac{16}{27}$ and $\frac{32}{81}$ to their numerators' fractional equivalents of $\frac{1}{2}$, $\frac{16}{32}$ and $\frac{32}{64}$.

13. **Answer (C):** Because each box is weighed two times, once with each of the other two boxes, the total $122 + 125 + 127 = 374$ pounds is twice the combined weight of the three boxes. The combined weight is $\frac{374}{2} = 187$ pounds.

14. **Answer (C):** There are only two possible spaces for the B in row 1 and only two possible spaces for the A in row 2. Once these are placed, the entries in the remaining spaces are determined.

The four arrangements are:

A	B	C
B	C	A
C	A	B

A	B	C
C	A	B
B	C	A

A	C	B
C	B	A
B	A	C

A	C	B
B	A	C
C	B	A

OR

The As can be placed either

A		
	A	
		A

or

A		
		A
	A	

In each case, the letter next to the top A can be B or C. At that point the rest of the grid is completely determined. So there are $2+2 = 4$ possible arrangements.

15. **Answer (B):** The sum of the points Theresa scored in the first 8 games is 37. After the ninth game, her point total must be a multiple of 9 between 37 and $37 + 9 = 46$, inclusive. The only such point total is $45 = 37 + 8$, so in the ninth game she scored 8 points. Similarly, the next point total must be a multiple of 10 between 45 and $45 + 9 = 54$. The only such point total is $50 = 45 + 5$, so in the tenth game she scored 5 points. The product of the number of points scored in Theresa's ninth and tenth games is $8 \cdot 5 = 40$.

16. **Answer (D):** The volume is $7 \times 1 = 7$ cubic units. Six of the cubes have 5 square faces exposed. The middle cube has no face exposed. So the total surface area of the figure is $5 \times 6 = 30$ square units. The ratio of the volume to the surface area is $7 : 30$.

OR

The volume is $7 \times 1 = 7$ cubic units. There are five unit squares facing each of six directions: front, back, top, bottom, left and right, for a total of 30 square units of surface area. The ratio of the volume to the surface area is $7 : 30$.

17. **Answer (D):** The formula for the perimeter of a rectangle is $2l + 2w$, so $2l + 2w = 50$, and $l + w = 25$. Make a chart of the possible widths, lengths, and areas, assuming all the widths are shorter than all the lengths.

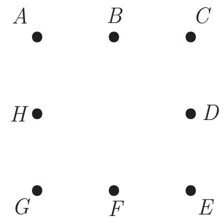
Width	1	2	3	4	5	6	7	8	9	10	11	12
Length	24	23	22	21	20	19	18	17	16	15	14	13
Area	24	46	66	84	100	114	126	136	144	150	154	156

The largest possible area is $13 \times 12 = 156$ and the smallest is $1 \times 24 = 24$, for a difference of $156 - 24 = 132$ square units.
Note: The product of two numbers with a fixed sum increases as the numbers get closer together. That means, given the same perimeter, the square has a larger area than any rectangle, and a rectangle with a shape closest to a square will have a larger area than other rectangles with equal perimeters.

18. **Answer (E):** The length of first leg of the aardvark's trip is $\frac{1}{4}(2\pi \times 20) = 10\pi$ meters. The third and fifth legs are each $\frac{1}{4}(2\pi \times 10) = 5\pi$ meters long. The second and sixth legs are each 10 meters long, and the length of the fourth leg is 20 meters. The length of the total trip is $10\pi + 5\pi + 5\pi + 10 + 10 + 20 = 20\pi + 40$ meters.

19. **Answer (B):** Choose two points. Any of the 8 points can be the first choice, and any of the 7 other points can be the second choice. So there are $8 \times 7 = 56$

ways of choosing the points in order. But each pair of points is counted twice, so there are $\frac{56}{2} = 28$ possible pairs.



Label the eight points as shown. Only segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HA}$ are 1 unit long. So 8 of the 28 possible segments are 1 unit long, and the probability that the points are one unit apart is $\frac{8}{28} = \frac{2}{7}$.

OR

Pick the two points, one at a time. No matter how the first point is chosen, exactly 2 of the remaining 7 points are 1 unit from this point. So the probability of the second point being 1 unit from the first is $\frac{2}{7}$.

20. **Answer (B):** Because $\frac{2}{3}$ of the boys passed, the number of boys in the class is a multiple of 3. Because $\frac{3}{4}$ of the girls passed, the number of girls in the class is a multiple of 4. Set up a chart and compare the number of boys who passed with the number of girls who passed to find when they are equal.

Total boys	Boys passed
3	2
6	4
9	6

Total girls	Girls passed
4	3
8	6

The first time the number of boys who passed equals the number of girls who passed is when they are both 6. The minimum possible number of students is $9 + 8 = 17$.

OR

Because $\frac{2}{3}$ of the boys passed, the number of boys who passed must be a multiple of 2. Because $\frac{3}{4}$ of the girls passed, the number of girls who passed must be a multiple of 3. Because the same number of boys and girls passed, the smallest possible number is 6, the least common multiple of 2 and 3. If 6 of 9 boys and 6 of 8 girls passed, there are 17 students in the class, and that is the minimum number possible.

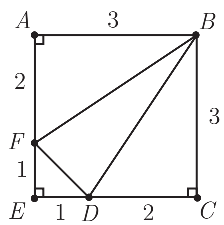
OR

Let $G =$ the number of girls and $B =$ the number of boys. Then $\frac{2}{3}B = \frac{3}{4}G$, so $8B = 9G$. Because 8 and 9 are relatively prime, the minimum number of boys and girls is 9 boys and 8 girls, for a total of $9 + 8 = 17$ students.

21. **Answer (C):** Using the formula for the volume of a cylinder, the bologna has volume $\pi r^2 h = \pi \times 4^2 \times 6 = 96\pi$. The cut divides the bologna in half. The half-cylinder will have volume $\frac{96\pi}{2} = 48\pi \approx 151\ \text{cm}^3$.
Note: The value of π is slightly greater than 3, so to estimate the volume multiply $48(3) = 144\ \text{cm}^3$. The product is slightly less than and closer to answer C than any other answer.

22. **Answer (A):** Because $\frac{n}{3}$ is at least 100 and is an integer, n is at least 300 and is a multiple of 3. Because $3n$ is at most 999, n is at most 333. The possible values of n are $300, 303, 306, \ldots, 333 = 3 \cdot 100, 3 \cdot 101, 3 \cdot 102, \ldots, 3 \cdot 111$, so the number of possible values is $111 - 100 + 1 = 12$.

23. **Answer (C):** Because the answer is a ratio, it does not depend on the side length of the square. Let $AF = 2$ and $FE = 1$. That means square $ABCE$ has side length 3 and area $3^2 = 9$ square units. The area of $\triangle BAF$ is equal to the area of $\triangle BCD = \frac{1}{2} \cdot 3 \cdot 2 = 3$ square units. Triangle DEF is an isosceles right triangle with leg lengths $DE = FE = 1$. The area of $\triangle DEF$ is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ square units. The area of $\triangle BFD$ is equal to the area of the square minus the areas of the three right triangles: $9 - (3 + 3 + \frac{1}{2}) = \frac{5}{2}$. So the ratio of the area of $\triangle BFD$ to the area of square $ABCE$ is $\dfrac{\frac{5}{2}}{9} = \dfrac{5}{18}$.



24. **Answer (C):** There are $10 \times 6 = 60$ possible pairs. The squares less than 60 are 1, 4, 9, 16, 25, 36 and 49. The possible pairs with products equal to the given squares are $(1,1)$, $(2,2)$, $(1,4)$, $(4,1)$, $(3,3)$, $(9,1)$, $(4,4)$, $(8,2)$, $(5,5)$, $(6,6)$ and $(9,4)$. So the probability is $\frac{11}{60}$.

25. **Answer (A):**

Circle #	Radius	Area
1	2	4π
2	4	16π
3	6	36π
4	8	64π
5	10	100π
6	12	144π

The total black area is $4\pi + (36 - 16)\pi + (100 - 64)\pi = 60\pi$ in^2.
So the percent of the design that is black is $100 \times \frac{60\pi}{144\pi} = 100 \times \frac{5}{12}$ or about 42%.