The MATHEMATICAL ASSOCIATION OF AMERICA
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# AMC 8

**(American Mathematics Contest 8)**

# Solutions Pamphlet

**Tuesday, NOVEMBER 19, 2002**

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the copyright law.

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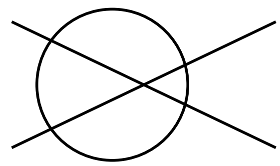
1. **(D)** Two distinct lines can intersect in one point whereas a line can intersect a circle in two points. The maximum number 5 can be achieved if the lines and circle are arranged as shown. Note that the lines could also meet outside the circle for the same result. (Other arrangements of the lines and circle can produce 0, 1, 2, 3, or 4 points of intersection.)



2. **(A)** Since the total \$17 is odd, there must be an odd number of \$5 bills. One \$5 bill plus six \$2 bills is a solution, as is three \$5 bills plus one \$2 bill. Five \$5 bills exceeds \$17, so these are the only two combinations that work.

3. **(C)** The smallest average will occur when the numbers are as small as possible. The four smallest distinct positive even integers are 2, 4, 6, and 8 and their average is 5.

   **Note:** These numbers form an arithmetic sequence. The average of the numbers in any arithmetic sequence is the average of the first and last terms.

4. **(B)** The next palindrome is 2112. The product of its digits is $2 \cdot 1 \cdot 1 \cdot 2 = 4$.

5. **(C)** Since 706 days is 700 plus 6 days, it is 100 weeks plus 6 days. Friday is 6 days after Saturday.

6. **(A)** Initially, volume increases with time as shown by graphs $A$, $C$, and $E$. But once the birdbath is full, the volume remains constant as the birdbath overflows. Only graph $A$ shows both features.

7. **(E)** There are $6 + 8 + 4 + 2 + 5 = 25$ students. Of the 25 students 5 prefer candy $E$ and $\frac{5}{25} = \frac{20}{100} = 20\%$.

8. **(D)** There are 15 French stamps and 9 Spanish stamps issued in the '80s. So there are $15 + 9 = 24$ European stamps listed in the table in the '80s.

9. **(B)** His South American stamps issued before the '70s include $4 + 7 = 11$ from Brazil that cost $11 \times \$0.06 = \$0.66$ and $6 + 4 = 10$ from Peru that cost $10 \times \$0.04 = \$0.40$. Their total cost is $\$0.66 + \$0.40 = \$1.06$.

10. **(E)** The '70s stamps cost: Brazil, $12(\$0.06) = \$0.72$; Peru, $6(\$0.04) = \$0.24$; France, $12(\$0.06) = \$0.72$; Spain, $13(\$0.05) = \$0.65$. The total is \$2.33 for the 43 stamps and the average price is $\frac{\$2.33}{43} \approx \$0.054 \approx 5.5$¢.

11. **(C)** To build the second square from the first, add 3 tiles. To build the third from the second, add 5 tiles. The pattern of adding an odd number of tiles continues. For the fourth square, add 7; for the fifth, add 9; for the sixth, add 11 and for the seventh, add 13.

   **OR**

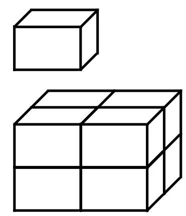
   The number of additional tiles needed is $7^2 - 6^2 = 49 - 36 = 13$.

**OR**

To build the second square from the first, add $2 + 1 = 3$ tiles. To build the third from the second, add $3 + 2 = 5$ tiles. The pattern continues and $7 + 6 = 13$.

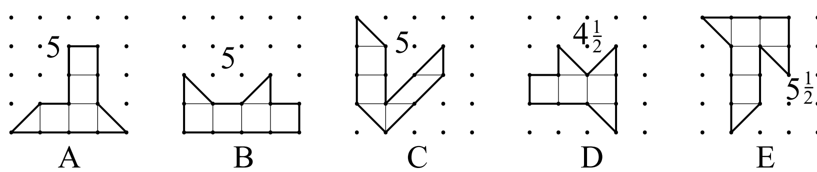
12. **(B)** Since the sum of the three probabilities is 1, the probability of stopping on region $C$ is $1 - \frac{1}{3} - \frac{1}{2} = \frac{6}{6} - \frac{2}{6} - \frac{3}{6} = \frac{1}{6}$.

13. **(E)** Since the exact dimensions of Bert's box do not matter, assume the box is $1 \times 2 \times 3$. Its volume is 6. Carrie's box is $2 \times 4 \times 6$, so its volume is 48 or 8 times the volume of Bert's box. Carrie has approximately $8(125) = 1000$ jellybeans.
**Note:** Other examples may help to see that the ratio is always 8 to 1.



14. **(B)** The first discount means that the customer will pay 70% of the original price. The second discount means a selling price of 80% of the discounted price. Because $0.80(0.70) = 0.56 = 56\%$, the customer pays 56% of the original price and thus receives a 44% discount.

15. **(E)** Areas may be found by dividing each polygon into triangles and squares as shown.




**Note:** Pick's Theorem may be used to find areas of geoboard polygons. If $I$ is the number of dots inside the figure, $B$ is the number of dots on the boundary and $A$ is the area, then $A = I + \frac{B}{2} - 1$. Geoboard figures in this problem have no interior points, so the formula simplifies to $A = \frac{B}{2} - 1$. For example, in polygon $D$ the number of boundary points is 11 and $\frac{11}{2} - 1 = 4\frac{1}{2}$.

16. **(E)** The areas are $W = \frac{1}{2}(3)(4) = 6$, $X = \frac{1}{2}(3)(3) = 4\frac{1}{2}$, $Y = \frac{1}{2}(4)(4) = 8$ and $Z = \frac{1}{2}(5)(5) = 12\frac{1}{2}$. Therefore, (E) is correct. $X + Y = 4\frac{1}{2} + 8 = 12\frac{1}{2} = Z$. The other choices are incorrect.

**OR**

By the Pythagorean Theorem, if squares are constructed on each side of any right triangle, the sum of the areas of the squares on the legs equal the area of the square on the hypotenuse. So $2X + 2Y = 2Z$, and $X + Y = Z$.

17. **(C)** Olivia solved at least 6 correctly to have scored over 25. Her score for six correct would be $6(+5) + 4(-2) = 22$, which is too low. If she answered 7 correctly, her score would be $7(+5) + 3(-2) = 29$, and this was her score. The correct choice is (C).

**OR**

Make a table:

| Right | + score | Wrong | - score | Total |
|---|---|---|---|---|
| 9 | 45 | 1 | $-2$ | 43 |
| 8 | 40 | 2 | $-4$ | 36 |
| 7 | 35 | 3 | $-6$ | 29 |

**OR**

If Olivia answers $x$ problems correctly, then she answered $10 - x$ incorrectly and her score was

$$\begin{aligned} 5x - 2(10 - x) &= 29 \\ 5x - 20 + 2x &= 29 \\ 7x &= 49 \\ x &= 7 \end{aligned}$$

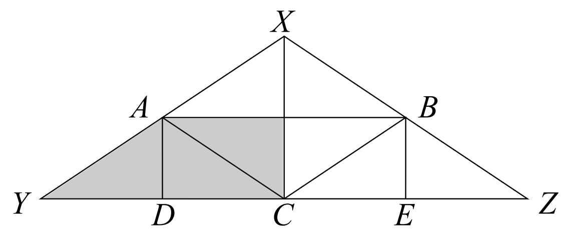
18. **(E)** In 5 days, Gage skated for $5 \times 75 = 375$ minutes, and in 3 days he skated for $3 \times 90 = 270$ minutes. So, in 8 days he skated for $375 + 270 = 645$ minutes. To average 85 minutes per day for 9 days he must skate $9 \times 85 = 765$ minutes, so he must skate $765 - 645 = 120$ minutes $= 2$ hours the ninth day.

**OR**

For 5 days Gage skated 10 minutes under his desired average, and for 3 days he skated 5 minutes over his desired average. So, on the ninth day he needs to make up $5 \times 10 - 3 \times 5 = 50 - 15 = 35$ minutes. To do this, on the ninth day he must skate for $85 + 35 = 120$ minutes $= 2$ hours.

19. **(D)** Numbers with exactly one zero have the form _0_ or _ _0, where the blanks are not zeros. There are $(9 \cdot 1 \cdot 9) + (9 \cdot 9 \cdot 1) = 81 + 81 = 162$ such numbers.

20. **(D)** Segments $\overline{AD}$ and $\overline{BE}$ are drawn perpendicular to $\overline{YZ}$. Segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ divide $\triangle XYZ$ into four congruent triangles. Vertical line segments $\overline{AD}$, $\overline{XC}$ and $\overline{BE}$ divide each of these in half. Three of the eight small triangles are shaded, or $\frac{3}{8}$ of $\triangle XYZ$. The shaded area is $\frac{3}{8}(8) = 3$.




**OR**

Segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ divide $\triangle XYZ$ into four congruent triangles, so the area of $\triangle XAB$ is one-fourth the area of $\triangle XYZ$. That makes the area of trapezoid $ABZY$ three-fourths the area of $\triangle XYZ$. The shaded area is one-half the area of trapezoid $ABZY$, or three-eighths the area of $\triangle XYZ$, and $\frac{3}{8}(8) = 3$.

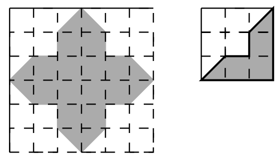
21. **(E)** There are 16 possible outcomes: $HHHH$, $HHHT$, $HHTH$, $HTHH$, $THHH$, $HHTT$, $HTHT$, $HTTH$, $THTH$, $THHT$, $TTHH$ and $HTTT$, $THTT$, $TTHT$, $TTTH$, $TTTT$. The first eleven have at least as many heads as tails. The probability is $\frac{11}{16}$.

22. **(C)** When viewed from the top and bottom, there are 4 faces exposed; from the left and right sides, there are 4 faces exposed and from the front and back, there are 5 faces exposed. The total is $4 + 4 + 4 + 4 + 5 + 5 = 26$ exposed faces.

**OR**

Before the cubes were glued together, there were $6 \times 6 = 36$ faces exposed. Five pairs of faces were glued together, so $5 \times 2 = 10$ faces were no longer exposed. This leaves $36 - 10 = 26$ exposed faces.

23. **(B)** The $6 \times 6$ square in the upper left-hand region is tessellated, so finding the proportion of darker tiles in this region will answer the question. The top left-hand corner of this region is a $3 \times 3$ square that has $3 + 2\left(\frac{1}{2}\right) = 4$ darker tiles. So $\frac{4}{9}$ of the total area will be made of darker tiles.



24. **(B)** Use 6 pears to make 16 oz of pear juice and 6 oranges to make 24 oz of orange juice for a total of 40 oz of juice. The percent of pear juice is $\frac{16}{40} = \frac{4}{10} = 40\%$.

**OR**

Miki can make 8 oz of orange juice with 2 oranges, so she can make 4 oz of orange juice with 1 orange. She can make 8 oz of pear juice from 3 pears, so she can make $\frac{8}{3}$ oz of pear juice from 1 pear. With 1 orange and 1 pear, she can make $4 + \frac{8}{3} = \frac{20}{3}$ oz of the blend, of which $\frac{8}{3}$ oz is pear juice. As a percent, $\frac{\frac{8}{3}}{\frac{20}{3}} = \frac{8}{20} = \frac{4}{10} = 40\%$ of the blend is pear juice.

25. **(B)** Only the fraction of each friend's money is important, so we can assume any convenient amount is given to Ott. Suppose that each friend gave Ott \$1. If this is so, then Moe had \$5 orignally and now has \$4, Loki had \$ 4 and now has \$3, and Nick had \$3, and now has \$2. The four friends now have $\$4 + \$3 + \$2 + \$3 = \$12$, so Ott has $\frac{3}{12} = \frac{1}{4}$ of the group's money. This same reasoning applies to any amount of money.

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