AMERICAN MATHEMATICS COMPETITIONS

AJHSME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

12th ANNUAL
AMERICAN JUNIOR HIGH SCHOOL MATHEMATICS EXAMINATION
(AJHSME)

THURSDAY, NOVEMBER 21, 1996

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. *However, photocopying this material is a violation of the copyright.*

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1. **(B)** The positive factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36. Of these, 4, 12, and 36 are also multiples of 4.

OR

List the multiples of 4 through 36, then mark those which are factors of 36:

$$\boxed{4},\ 8,\ \boxed{12},\ 16,\ 20,\ 24,\ 28,\ 32,\ \boxed{36}.$$

OR

Since $36 = 2^2 3^2$, its positive factors are:

$$\begin{array}{lll} 2^0 3^0, & 2^1 3^0, & \boxed{2^2 3^0}, \\ 2^0 3^1, & 2^1 3^1, & \boxed{2^2 3^1}, \\ 2^0 3^2, & 2^1 3^2, & \boxed{2^2 3^2}. \end{array}$$

Of these, only three are also multiples of $4 = 2^2$.

2. **(C)** Starting with 10:
   - José computes, consecutively, 9, then 18, and finally 20.
   - Thuy computes, consecutively, 20, then 19, and finally 21.
   - Kareem computes, consecutively, 9, then 11, and finally 22.

   Thus Kareem gets the largest final answer.

Note. Any number n could have been used instead of 10 to obtain the same result.

	José	**Thuy**	**Kareem**
Start :	n	n	n
First :	$n-1$	$2n$	$n-1$
Then :	$2(n-1) = 2n-2$	$2n-1$	$(n-1)+2 = n+1$
Finally :	$(2n-2)+2 = \underline{2n}$	$(2n-1)+2 = \underline{2n+1}$	$2(n+1) = \underline{2n+2}$

Since $2n+2 > 2n+1 > 2n$, Kareem gets the largest final answer.

3. **(A)** The first row is $1, 2, 3, \ldots, 7, 8$ and the last row is $57, 58, 59, \ldots, 63, 64$. Thus the four corner numbers are 1, 8, 57, and 64, and their sum is 130.

OR

Listing the array yields:

1	2	3	4	5	6	7	**8**
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	**64**

The sum of the numbers in the corners is 130.

4. **(B)** $\dfrac{2+4+6+\cdots+34}{3+6+9+\cdots+51}=\dfrac{2(1+2+3+\cdots+17)}{3(1+2+3+\cdots+17)}=\dfrac{2}{3}$.

5. **(A)** Examine the sign of each choice. Since P is to the left of Q, $P-Q$ is negative. All the other answers are positive:
   - Since P and Q are both negative, their product **(B)** is positive.
   - Since S and Q are of opposite signs, their quotient is negative. When this quotient is multiplied by a negative number, P, the result **(C)** is positive.
   - Since R and $P \cdot Q$ are both positive, their quotient **(D)** is positive.
   - Since $S+T$ and R are both positive, their quotient **(E)** is positive.

6. **(C)** To obtain the smallest result, use the three smallest numbers. This yields three choices:

$$3(5+7)=36, \qquad 5(3+7)=50, \quad \text{and} \quad 7(3+5)=56.$$

Thus 36 is the smallest result.

OR

Since multiplication is repeated addition, it follows that the smallest result should use the smallest number as the multiplier and the other two of the three smallest numbers for the sum. Thus $3(5+7)=36$ is the smallest result.

7. **(B)** Make a table.

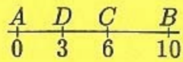
Month:	0	1	2	3	4	5
Brent:	4	16	64	256	1024	4096
Gretel:	128	256	512	1024	2048	4096

Thus it takes 5 months for the number of goldfish to be equal.

OR

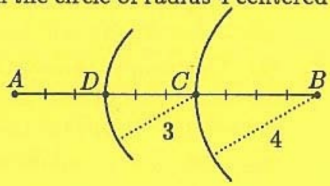
The ratio of the number of Gretel's goldfish to the number of Brent's goldfish decreases by a factor of 2 every month. Because they initially differ by a factor of $128/4 = 32 = 2^5$, this indicates that the process will take 5 months.

8. **(B)** The shortest distance occurs when the four points are collinear. Then $BC = 4$ and $CD = 3$ so that $AD = 3$ as shown.



OR

Once A and B are chosen, C can be anywhere on the circle of radius 4 centered at B. The closest point to A would be on the segment between B and A. Then D could be anywhere on the circle of radius 3 centered at C. Once again the closest point to A is on the line segment between C and A. Thus the distance from A to D is 3 units.

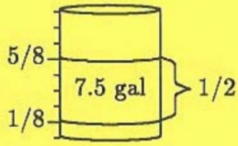


9. **(D)** To find the number, divide 2 by 5 to obtain 2/5. The reciprocal of 2/5 is 5/2, and 100 times 5/2 equals 250.

OR

Let n be the unknown number. Then $5 \cdot n = 2$ so $n = 2/5$. The reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$, and $100 \cdot \frac{5}{2} = 250$.

10. **(D)** The gasoline tank going from $1/8$ to $5/8$ full represents an increase of $5/8 - 1/8 = 1/2$ tank. Since half a tank is 7.5 gallons, it follows that a full tank is $2 \times 7.5 = 15$ gallons.



11. **(D)** Since x is near zero, $3 + x$ and $3 - x$ are near 3. Also $3 \cdot x$ and $x/3$ are near zero. However, $3/x$ is the number

$$3\underbrace{0000\ldots0000}_{\text{1997 zeros}},$$

which is a 3 followed by 1997 zeros. This number is much larger than any of the other alternatives.

12. **(B)** The sum of the eleven numbers is 66. For the average of ten numbers to be 6.1, the sum of the ten numbers must be $10 \times 6.1 = 61$. Thus, remove the 5.

13. **(E)** Since 50% of 800 is 400,

the organizers expect $800 + 400 = 1200$ participants in 1997.
Since 50% of 1200 is 600, $1200 + 600 = 1800$ are expected in 1998.
Since 50% of 1800 is 900, $1800 + 900 = 2700$ are expected in 1999.

OR

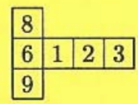
Make a table.

YEAR	NUMBER OF PARTICIPANTS
1996	800
1997	$800 + (0.5 \times 800) = 1200$
1998	$1200 + (0.5 \times 1200) = 1800$
1999	$1800 + (0.5 \times 1800) = 2700$

OR

An increase of 50% means there will be $100\% + 50\%$ or 1.5 times as many participants in each successive year. Thus in 1999, three years from 1996, there will be $800 \times 1.5^3 = 2700$ participants.

14. **(B)** Determine that one suitable arrangement of the digits is as indicated, and then compute



$$8 + 6 + 9 + 1 + 2 + 3 = 29.$$

OR

Since $7 + 8 + 9 = 24$, the digits in the column must be 6, 8, and 9 in some order. The sum of the digits in the row that are not also in the column must be at least $1 + 2 + 3 = 6$. Then the square common to both the column and the row contains at most $12 - (1 + 2 + 3) = 12 - 6 = 6$. Therefore, the 8 or 9 cannot be used in the common square, and hence that square must contain the digit 6. Thus the digits are 1, 2, 3, 6, 8, and 9, which have a sum of 29.

15. **(E)** The last digit of a product is determined by the product of the last digits of the factors. Since $2 \cdot 6 \cdot 2 \cdot 6 = 144$, the last digit of the product is 4. Since multiples of 5 end in 0 or 5, any number with last digit 4 leaves a remainder of 4 when divided by 5.

16. **(C)** Combining in groups of four yields

$$1-2-3+4=0, \quad 5-6-7+8=0, \quad 9-10-11+12=0,$$

and so on. Since there are 499 groups of four in 1996, it follows that the sum is zero.

OR

Combining in pairs yields

$$\underbrace{\underbrace{(1-2)+(-3+4)}_{(-1)+1}}_{0} + \underbrace{\underbrace{(5-6)+(-7+8)}_{(-1)+1}}_{0} + \cdots + \underbrace{\underbrace{(1993-1994)+(-1995+1996)}_{(-1)+1}}_{0}$$

so the sum is 0.

17. **(C)** Since $OPQR$ is a square and point Q has coordinates $(2,2)$, it follows that point P has coordinates $(2,0)$ and point R has coordinates $(0,2)$. Thus the side of square $OPQR$ is 2 and the area of the 2 by 2 square $OPQR$ is $2^2=4$. The area of triangle PQT is $(1/2)(PT)(PQ)=(1/2)(PT)(2)=PT$. Since the area of the triangle equals the area of the square, $PT=4$. Thus the point T has coordinates $(-2,0)$.

OR

If the area of square $OPQR$ and triangle PQT are equal, then the area of the small triangle with side $\overline{RQ}$ that is in the square but not in the triangle must equal the area of the small triangle with side $\overline{OT}$ that is in the triangle but not in the square. These two triangles will be the same size and shape when $\overline{QT}$ intersects $\overline{RO}$ at its midpoint and $OT=RQ=2$, so T must have coordinates $(-2,0)$.

18. **(A)** After the first change, Ana's salary was $\$2000+0.20(\$2000)=\$2400$. After the second change, Ana's salary was $\$2400-0.20(\$2400)=\$1920$.

OR

After the two changes, Ana's salary was 80% of 120% of \$2000, or $0.80(1.20)(\$2000)=\$1920$.

19. **(C)** The first pie chart shows that 22% of 2000, or 440 students at East prefer tennis. The second chart shows that 40% of 2500, or 1000 students at West prefer tennis. Thus 1440 of the total of 4500 students prefer tennis. This gives $1440/4500=0.32$, or 32% that prefer tennis.

OR

East has 2000 of the 4500 students, and West has 2500 of the 4500 students. Using a weighted average yields $\frac{2000}{4500} \cdot 0.22 + \frac{2500}{4500} \cdot 0.40 = 0.32$, or 32%.

20. **(A)** After the special key is pressed once, the calculator display reads -0.25 since $1/(1-5) = 1/(-4) = -0.25$. If the key is pressed again, the calculator display reads 0.8 since $1/(1-(-0.25)) = 1/(1.25) = 0.8$. If the key is pressed a third time, the calculator display reads 5, since $1/(1-0.8) = 1/(0.2) = 5$. Thus pressing the special key three times returns to the original calculator display. The calculator display will continue to cycle through the three answers -0.25, 0.8, and 5. Since 100 is 1 more than a multiple of 3, the calculator display will be -0.25.

21. **(D)** The sum of three numbers is even if all three numbers are even, or if two numbers are odd and one is even. Since there are only two even numbers in the set, it follows that the three numbers must include two odd numbers and one even. The possibilities are:

$$\begin{array}{lll} \{89,95,132\} & \{89,99,132\} & \{89,173,132\} \\ \{89,95,166\} & \{89,99,166\} & \{89,173,166\} \\ \\ \{95,99,132\} & \{95,173,132\} & \{99,173,132\} \\ \{95,99,166\} & \{95,173,166\} & \{99,173,166\}. \end{array}$$

Thus there are 12 possibilities.

OR

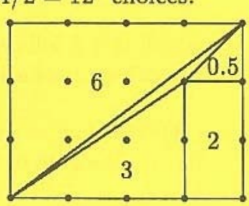
Let O stand for odd and E stand for even. The numbers given are O, O, O, E, E and O. There are only two ways that the sum of three numbers is even.

Case I: $E + E + E = E$, and

Case II: $O + O + E = E$.

Since there are only two even numbers, **Case I** cannot happen. For **Case II**, $O + O + E$, counting choices yields 4 choices for the first odd, 3 remaining choices for the second odd, and 2 choices for the even, for a total of $4 \times 3 \times 2 = 24$ choices. However, since $O_1 + O_2 + E = O_2 + O_1 + E$, the number of choices is reduced by a factor of 2. Hence there are $24/2 = 12$ choices.

22. **(B)** From the total area of 12, subtract the areas of the four surrounding polygons whose areas are indicated in the diagram. Thus the area of the remaining triangle ABC is



$$12 - 6 - 3 - 0.5 - 2 = 0.5 = 1/2.$$

Note. Points whose coordinates are integers are called *lattice* points. According to Pick's Theorem, if there are I lattice points in the interior of a triangle and B lattice points on the boundary, then the area of the triangle is $I + B/2 - 1$. In this problem, $I = 0$ and $B = 3$. Therefore the area of the triangle is $0 + 3/2 - 1 = 1/2$.

23. **(E)** Had there been \$5 more in the company fund, there would have been $\$95+\$5 = \$100$ which would have been enough to give each employee another \$5. Thus there are $\$100/\$5 = 20$ employees. So the company fund contained $20 \cdot \$45 + \$95 = \$995$.

OR

The \$95 left over represents \$5 for each employee who would have received \$50. Thus there are $95/5 = 19$ of these employees. Hence the amount of money in the company fund before any bonuses were paid was $(19 \times \$50) + (1 \times \$45) =$ \$995.

24. **(C)** Since the sum of the measures of the angles of a triangle is 180°, in triangle ABC it follows that

$$\angle BAC + \angle BCA = 180^\circ - 50^\circ = 130^\circ.$$

The measures of angles DAC and DCA are half that of angles BAC and BCA, respectively, so

$$\angle DAC + \angle DCA = \frac{130^\circ}{2} = 65^\circ.$$

In triangle ACD, we have $\angle ADC = 180^\circ - 65^\circ = 115^\circ$.

25. **(A)** Suppose that the circle has radius 1. Then, being closer to the center of the region than the boundary of the region (the circle) would mean the chosen point must be inside the circle of radius $1/2$ with the same center as the larger circle of radius 1. The area of the smaller region is $\pi(1/2)^2 = \pi/4$, and the area of the total region is $\pi(1)^2 = \pi$. Since the area of the smaller region is $1/4$ of the area of the total region, the required probability is $1/4$.

Note. The odds that the point is closer to the center are $1 : 3$.