AMERICAN MATHEMATICS COMPETITIONS

AJHSME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

10th ANNUAL
AMERICAN JUNIOR HIGH SCHOOL MATHEMATICS EXAMINATION
(AJHSME)

THURSDAY, NOVEMBER 17, 1994

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using material normally associated with the mathematics curriculum for the eighth grade or lower. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. *However, photocopying this material is a violation of the copyright.*

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1. **(D)** Express all the choices as fractions using their least common denominator:

$$\frac{1}{3}=\frac{8}{24},\quad \frac{1}{4}=\frac{6}{24},\quad \frac{3}{8}=\frac{9}{24},\quad \frac{5}{12}=\frac{10}{24},\quad \frac{7}{24}.$$

Thus, $5/12$ is the largest.

OR

Express each choice as a decimal:

$\frac{1}{3}=0.333\ldots$ $\frac{1}{4}=0.25$ $\frac{3}{8}=0.375$ $\frac{5}{12}=0.41666\ldots$ $\frac{7}{24}=0.291666\ldots$

Thus, $5/12$ is the largest.

OR

All choices are less than $1/2$, so the choice least distant from $1/2$ will be the largest:

fraction :	1/3	1/4	3/8	**5/12**	7/24
distance from 1/2 :	1/6	1/4	1/8	**1/12**	5/24

2. **(D)** The sum of all the numerators is 100. Consequently, the sum of all the fractions is $100/10=10$.

OR

Regroup the fractions before adding:

$$\left(\frac{1}{10}+\frac{9}{10}\right)+\left(\frac{2}{10}+\frac{8}{10}\right)+\left(\frac{3}{10}+\frac{7}{10}\right)+\left(\frac{4}{10}+\frac{6}{10}\right)+\left(\frac{5}{10}+\frac{55}{10}\right)=$$

$1+1+1+1+6=10.$

3. **(C)** Maria's working day ends 8 hours and 45 minutes later than 7:25 A.M. Eight hours later than 7:25 A.M. is 3:25 P.M. Forty-five minutes later than 3:25 P.M. is 4:10 P.M.

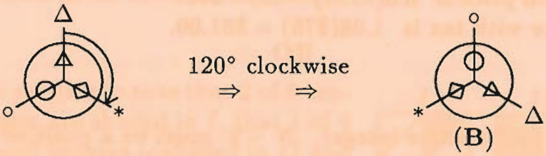
OR

7:25 A.M.	begin
+8:00	work time
+ :45	lunch time
15:70 = 16:10 = 4:10 P.M.	end

OR

Maria's total time at work including lunch is 15 minutes less than 9 hours. Nine hours later than 7:25 A.M. is 4:25 P.M. Fifteen minutes before 4:25 P.M. is 4:10 P.M.

4. **(B)**



5. **(B)** 1 mile $=$ 8 furlongs $= 8 \times (1 \text{ furlong}) = 8 \times (40 \text{ rods}) = 320$ rods.

OR

$$1\ \cancel{\text{mile}} \times \frac{8\ \cancel{\text{furlongs}}}{1\ \cancel{\text{mile}}} \times \frac{40 \text{ rods}}{1\ \cancel{\text{furlong}}} = (8 \times 40) \text{ rods} = 320 \text{ rods}.$$

6. **(A)** Any selection of six consecutive positive whole numbers has at least one multiple of 5 and at least one multiple of 2. Thus, the product has at least one factor of $5 \times 2 = 10$ and must end in 0.

OR

The statement of the problem implies that any six consecutive positive numbers can be used. Compute $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 120$ to find that the last digit is 0. Or, even easier, multiply any six consecutive integers containing 10 to see without multiplication that the last digit must be 0.

7. **(B)** Since the sum of the angles in any triangle is $180°$,

$$\angle ABE = 180° - (60° + 40°) = 80°.$$

Since $\angle ABD$ and $\angle DBC$ together form a straight angle, their sum is $180°$, so $\angle DBC = 180° - 80° = 100°$. Thus $\angle BDC = 180° - (100° + 30°) = 50°$.

8. **(C)** Since the sum of the three digits must equal 25 and $8 + 8 + 8 = 24$, it follows that at least one of the digits must be a 9. If only one digit is 9, then the other two digits add up to 16 and are each 8. If two digits are 9, then the other digit is 7. There are six numbers that meet the requirement:

988, 898, 889, 997, 979, 799.

9. **(A)** The 20% discount lowers the price to \$80.
Then the \$5 coupon reduces the price to \$75.
The sales tax is 8% of \$75, or $0.08 \times \$75 = \$6$.
Thus, the final cost is $\$75 + \$6 = \$81.00$.

OR

The discounted price is $0.8(\$100) - \$5 = \$75$.
The final price with tax is $1.08(\$75) = \$81.00$.

10. **(A)** Since N is a positive integer, $N + 2$ must be a positive integer divisor of 36. There are 9 positive integer divisors of 36:

$$36, \quad 18, \quad 12, \quad 9, \quad 6, \quad 4, \quad 3, \quad 2, \quad 1.$$

Express these numbers in the form $N + 2$:

$$34{+}2, \quad 16{+}2, \quad 10{+}2, \quad 7{+}2, \quad 4{+}2, \quad 2{+}2, \quad 1{+}2, \quad 0{+}2, \quad -1{+}2.$$

Of these, seven have a positive value for N.

OR

Given $N > 0$, it follows that $N + 2 > 2$. Since $N + 2$ must be a divisor of 36, there are 7 such divisors greater than 2: $3, 4, 6, 9, 12, 18, 36$.

11. **(B)** Since the total number of girls was 48 and there were 20 girls from Jones, it follows that there were $48 - 20 = 28$ girls from Clay. The total number of students from Clay was 60. Thus, there were $60 - 28 = 32$ boys from Clay.

OR

Complete a table starting with the given information:

	B	G	Tot
JMS :		20	40
CMS :			60
Total :	52	48	100

	B	G	Tot
JMS :	$\boxed{20}$	20	40
CMS :			60
Total :	52	48	100

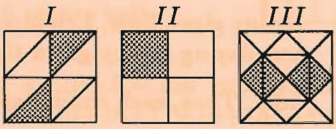
	B	G	Tot
JMS :	20	20	40
CMS :	$\boxed{32}$		60
Total :	52	48	100

There were 32 boys from Clay Middle School.

12. **(A)** The shaded area of square II is clearly $1/4$ of the total area. In square I each shaded triangle is $1/2$ of $1/4$ or $1/8$ of the total area, so the total shaded area is $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$ of the total. In square III, each shaded diamond can be visualized as two triangles. Each of these triangles is $1/4$ of $1/4$ or $1/16$ of the total area, so the four triangles make up $4/16$ or $1/4$ of the total. Thus the shaded areas in all three are equal.

OR

Partition the squares to note that 2 of 8 congruent triangles are shaded in I, that 1 of 4 congruent small squares is shaded in II and that 4 of 16 congruent triangles are shaded in III.



13. **(C)** Since $\frac{1}{6}=\frac{2}{12}=\frac{4}{24}$ and $\frac{1}{4}=\frac{3}{12}=\frac{6}{24}$, it follows that the number halfway between $\frac{1}{6}$ and $\frac{1}{4}$ is $\frac{5}{24}$.

OR

The number halfway between any two numbers is their arithmetic mean (average):

$$\frac{\frac{1}{6}+\frac{1}{4}}{2}=\frac{\frac{4+6}{24}}{2}=\frac{\frac{10}{24}}{2}=\frac{5}{24}.$$

OR

Estimating each fraction to three decimal places and calculating the average yields

$$\frac{\frac{1}{6}+\frac{1}{4}}{2}\approx\frac{0.167+0.250}{2}=\frac{0.417}{2}=0.2085$$

which is closest to $\frac{5}{24}\approx 0.2083$.

14. **(E)** Two children are playing at the same time, so there are $2\times 90=180$ minutes of playing time. This must be divided equally 5 ways, so each child gets $180/5=36$ minutes.

OR

If pairball were played by one person at a time, each child would get $1/5$ of 90, or 18 minutes of playing time. Since two players are required, each child gets double this, or 36 minutes.

OR

There are 10 ways to pair the children, so each session lasts $90/10 = 9$ minutes. Each child is paired with four other children, so each child will play for $4 \times 9 = 36$ minutes.

Query. What are the 10 ways to pair the five children?

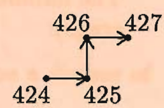
OR

At any given time, 2 children are playing and 3 are not, so $2/5$ of the children are playing at any given time. This implies that each child plays $2/5$ of the time. Thus, each child plays $\frac{2}{5} \times 90 = 36$ minutes.

15. **(A)** The path repeats every four numbers. Since 425 leaves a remainder of 1 when divided by 4 and 427 leaves a remainder of 3 when divided by 4, it follows that the pattern from point 425 to point 427 is the same as that from point 1 to point 3.

OR

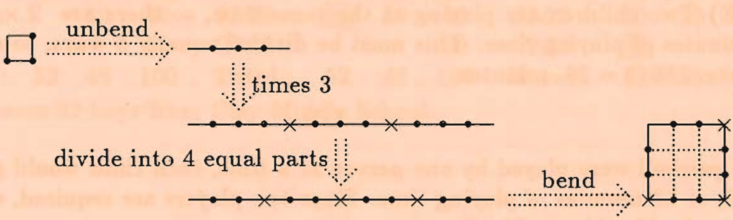
The pattern starts over every four numbers, that is, at 0, 4, 8, etc. All the multiples of 4 are in the same position in their section of the pattern, so draw the portion of the diagram using the multiple of 4 just less than 425.



16. **(E)** The perimeter being 3 times larger implies that a side of the larger square is 3 times a side of the smaller square. Thus, since the area of a square is the length of the side squared, it follows that the area of the larger square will be $3^2 = 9$ times the area of the smaller square.

OR

The sketch shows that when the perimeter is tripled, it encloses 9 squares equal to the original square:



OR

Since the ratio of areas of similar figures is the square of the ratio of any matching linear part, it follows that the ratio of the areas is $\left(\frac{3}{1}\right)^2 = \frac{9}{1}$.

OR

Use a sample case: Area of a 1 by 1 square is 1 square unit.
Area of a 3 by 3 square is 9 square units.

17. **(D)** The volume of the snow on the driveway is $4 \times 10 \times 3 = 120$ cubic yards. Adding the rates for 7 consecutive hours yields $20+19+18+17+16+15+14 = 119$, while 8 consecutive hours yields $20+19+18+17+16+15+14+13 = 132$. The 7-hour solution is closest to 120 cubic yards.

OR

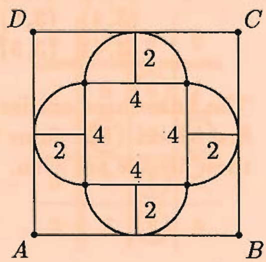
The volume of the snow on the driveway is $4 \times 10 \times 3 = 120$ cubic yards. Computing the amount left after each hour, we have:

$$120 - 20 = 100, \quad 100 - 19 = 81, \quad 81 - 18 = 63, \quad 63 - 17 = 46,$$
$$46 - 16 = 30, \quad 30 - 15 = 15, \quad 15 - 14 = 1, \quad 1 - 13 = -12.$$

Since 1 is closer to zero than is -12, the 7-hour solution is closest to 120 cubic yards.

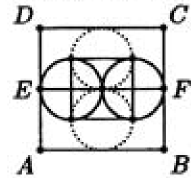
18. **(B)** Both graphs **(A)** and **(B)** show a flat segment indicating no change in distance, or a stop. Graph **(A)** shows a constant change in distance indicating a constant rate of driving before and after the stop. Graph **(B)** shows a slow change in distance (shallow graph) followed by a more rapid change in distance (steep graph) indicating a slower rate followed by a faster rate of driving before the stop. After the stop, it shows a faster rate of driving followed by a slower rate. Thus, graph **(B)** corresponds to Mike's trip.

19. **(E)** The radius of each semicircle is 2, since it is 1/2 the length of a side of the 4 by 4 square. Since the length of a side of $ABCD$ is the length of a side of the 4 by 4 square plus two radii of semicircles [see figure], each side of $ABCD$ measures $4 + 2(2) = 8$, so the area of $ABCD$ is $8^2 = 64$.



OR

Complete each semicircle to a circle, and note that since the radius of each circle is half the side-length of the smaller square, the four circles must intersect at the center of both squares. Thus, $AB = EF = 8$ since the length of $\overline{EF}$ is the length of two diameters. Hence, the area of $ABCD$ is $8^2 = 64$.



20. **(D)** Small numerators and large denominators yield small fractions. Use 1 and 2 for numerators and 8 and 9 for denominators to obtain the smallest fractions, then compare the sums

$$\frac{1}{8}+\frac{2}{9}=\frac{9+16}{72}=\frac{25}{72} \quad \text{and} \quad \frac{1}{9}+\frac{2}{8}=\frac{8+18}{72}=\frac{26}{72}$$

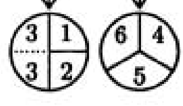
to see that $\frac{25}{72}$ is the answer.

Note. Analyzing the sums yields $\frac{1}{8}+\frac{1}{9}+\frac{1}{9}$ which is smaller than $\frac{1}{9}+\frac{1}{8}+\frac{1}{8}$.

21. **(C)** It is *possible* to get four gumballs of the same color by buying 4, 5, 6, 7, 8, or 9. However, the first nine gumballs *might* consist of three of each color, so nine gumballs will *not guarantee* four of the same color. In this case, the tenth gumball must match one of the previous colors, giving four of that color.

22. **(D)** Subdivide the 3-space on the first wheel so that wheel is divided into four equal regions. Each of the four regions has the same probability of occurring when the first wheel is spun. Hence, the sample space is

(1,4)	(1,5)	(1,6)
(2,4)	(2,5)	(2,6)
(3,4)	(3,5)	(3,6)
(3,4)	(3,5)	(3,6)



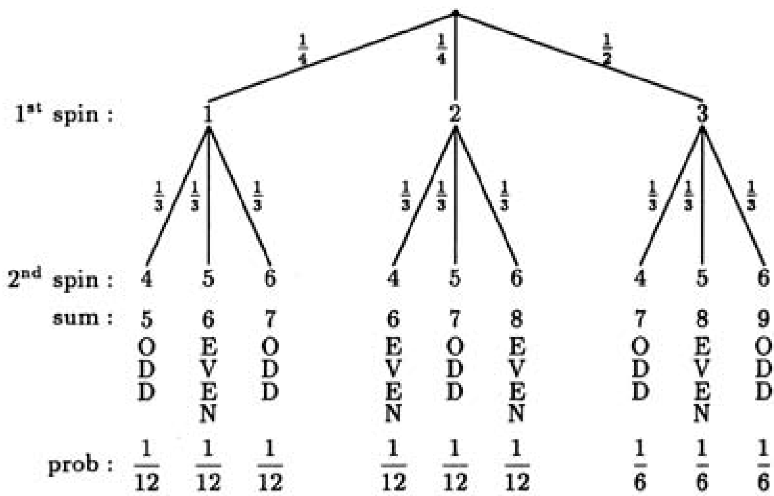
The final three entries were repeated to show the double space for 3 on the first wheel. The sums for these entries are $5,6,7,6,7,8,7,8,9,7,8,9$. Five of these twelve are even.

OR

There are two ways to get an even sum, (odd + odd) or (even + even). The first outcome happens if one spins 1 or 3 on the first wheel (3 chances out of 4) and 5 on the second wheel (1 chance out of 3) for a probability of $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$. The second outcome happens if one spins a 2 on the first wheel (1 chance out of 4) and 4 or 6 on the second wheel (2 chances out of 3) for a probability of $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$. Thus the total probability of an even sum is $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

OR

Use a probability tree diagram:



Thus, the probability that the sum is even is $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{5}{12}$.

OR

List equally likely outcomes in a table. Even sums are marked. The probability is 5/12.

+	4	5	6
1	5	(6)	7
2	(6)	7	(8)
3	7	(8)	9
3	7	(8)	9

OR

$$P(\text{even}) = P(3+5) + P(1+5) + P(2+4) + P(2+6)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{5}{12}.$$

23. **(D)** Since the sum must be only three digits, it follows that $X \neq 9$ because $X = 9$ would give a four-digit sum. Thus the largest value for X must be 8. Then, to obtain the largest sum, $Y = 9$. This yields

$$\begin{array}{r} 888 \\ 98 \\ +\ \ 8 \\ \hline 994 \end{array}$$

Thus, the largest sum has the form YYZ.

24. **(B)** If there is any square painted green, then all the squares above or to the right of it must also be green. Therefore, the possible patterns of colors are:

$$\begin{bmatrix} R & R \\ R & R \end{bmatrix} \quad \begin{bmatrix} R & G \\ R & R \end{bmatrix} \quad \begin{bmatrix} G & G \\ R & R \end{bmatrix}$$
$$\begin{bmatrix} R & G \\ R & G \end{bmatrix} \quad \begin{bmatrix} G & G \\ R & G \end{bmatrix} \quad \begin{bmatrix} G & G \\ G & G \end{bmatrix}$$

Thus, there are 6 different ways to paint the squares according to the requirements of the problem.

OR

All the green squares on any row must be to the right. The number of green squares in any row must be at least as large as the number of green squares in any lower row. Therefore, the number of ways to paint n squares green is the number of sums $a + b = n$ with $2 \geq a \geq b \geq 0$, and n can be any number from 0 through 4. There are 6 such sums:

$$\begin{array}{ccc} 0+0=0 & 1+0=1 & 2+0=2 \\ 1+1=2 & 2+1=3 & 2+2=4 \end{array}$$

Therefore, there are 6 ways to paint the 2 by 2 square according to the requirements of the problem.

Query. What if the original square were 3 by 3? 4 by 4?

25. **(A)** Since $\underbrace{9999\cdots99}_{\text{94 nines}} = 1\underbrace{0000\cdots00}_{\text{94 zeros}} - 1$, we have

$$\underbrace{9999\cdots99}_{\text{94 nines}} \times \underbrace{4444\cdots44}_{\text{94 fours}} = (1\underbrace{0000\cdots00}_{\text{94 zeros}} - 1) \times \underbrace{4444\cdots44}_{\text{94 fours}}$$

$$= (\underbrace{4444\cdots44}_{\text{94 fours}}\underbrace{0000\cdots00}_{\text{94 zeros}} - \underbrace{4444\cdots44}_{\text{94 fours}}$$

which is

$$\begin{array}{r} 4444\cdots44\,0000\cdots00 \\ -\ 4444\cdots44 \\ \hline \underbrace{4444\cdots4}_{\text{93 fours}}3\underbrace{5555\cdots5}_{\text{93 fives}}6 \end{array}$$

The sum of the digits of this answer is

$$93(4) + 3 + 93(5) + 6 = 93(4+5) + (3+6) = 94(9) = 846.$$

OR

Try smaller cases to observe a pattern:

$$\begin{array}{ccccr} 9 \times 4 & = & 36 & \text{sum} = & 9 = 9 \times 1 \\ 99 \times 44 & = & 4356 & & = 18 = 9 \times 2 \\ 999 \times 444 & = & 443556 & & = 27 = 9 \times 3 \\ 9999 \times 4444 & = & 44435556 & & = 36 = 9 \times 4 \\ \vdots & = & \vdots & & \\ \underbrace{9999\cdots99}_{\text{94 nines}} \times \underbrace{4444\cdots44}_{\text{94 fours}} & = & \underbrace{4444\cdots4}_{\text{93 fours}}3\underbrace{5555\cdots5}_{\text{93 fives}}6 & & = 9 \times 94 \end{array}$$

The sum of the digits of this answer is

$$93(4) + 3 + 93(5) + 6 = 93(4+5) + (3+6) = 94(9) = 846.$$

Query. What is the sum of the digits when any 94-digit number is multiplied by $\underbrace{9999\cdots99}_{\text{94 nines}}$?