AMERICAN MATHEMATICS COMPETITIONS

# AJHSME SOLUTIONS PAMPHLET
## FOR STUDENTS AND TEACHERS

## 9th ANNUAL

# AMERICAN JUNIOR HIGH SCHOOL MATHEMATICS EXAMINATION (AJHSME)

## THURSDAY, NOVEMBER 18, 1993

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using material normally associated with the mathematics curriculum for the eighth grade or lower. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. *However, photocopying this material is a violation of the copyright.*

Correspondence about the problems and solutions should be addressed to:


Mr Bruce Brombacher, AJHSME Chairman
Jones Middle School
Upper Arlington, OH 43221


Orders for prior year Examination questions and Solutions Pamphlets or Problem Books should be addressed to:


Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0658

1. **(C)** $\frac{1}{2} \times (-72) = -36.$

2. **(C)** $\frac{49}{84} = \frac{7 \times 7}{7 \times 12} = \frac{7}{12}.$
The sum of the numerator and the denominator is $7 + 12 = 19.$

3. **(B)** Factoring each number into prime factors yields

$$39 = 3 \times 13, \quad 51 = 3 \times 17, \quad 77 = 7 \times 11, \quad 91 = 7 \times 13 \quad \text{and} \quad 121 = 11 \times 11.$$

The largest of these prime factors is 17, which is a factor of 51.

4. **(E)**
$$\begin{aligned} 1000 \times 1993 \times 0.1993 \times 10 &= ((1000 \times 10) \times 0.1993) \times 1993 \\ &= (10,000 \times 0.1993) \times 1993 \\ &= 1993 \times 1993 = (1993)^2. \end{aligned}$$

5. **(C)** The unshaded area is half the total, and each of the shaded areas is one fourth of the total. This is represented in bar graph **(C)**.

6. **(B)** Three cans of soup are needed for 15 children, so the remaining 2 cans of soup will feed $2 \times 3 = 6$ adults.

7. **(A)** $3^3 + 3^3 + 3^3 = 3(3^3) = 3(3 \times 3 \times 3) = 3 \times 3 \times 3 \times 3 = 3^4.$

**OR**

$$3^3 + 3^3 + 3^3 = 27 + 27 + 27 = 81 = 9 \times 9 = 3 \times 3 \times 3 \times 3 = 3^4.$$

8. **(D)** Since she takes one half of a pill every other day, one pill will last 4 days. Hence 60 pills will last $60 \times 4 = 240$ days, or about 8 months.

9. **(D)** Substituting the values from the table yields

$$(2 * 4) * (1 * 3) = 3 * 3 = 4.$$

**Query.** Would evaluating the products

$$((2 * 4) * 1) * 3, \quad (2 * (4 * 1)) * 3, \quad 2 * ((4 * 1)) * 3) \quad \text{and} \quad 2 * (4 * (1 * 3)),$$

yield the same result?

10. **(B)** The graph shows the following changes in the price of the card:

| | | |
|---|---|---|
| *Jan* : | \$2.50 to \$2.00 | **drop of \$0.50** |
| *Feb* : | \$2.00 to \$4.00 | **rise of \$2.00** |
| *Mar* : | \$4.00 to \$1.50 | **drop of \$2.50** |
| *Apr* : | \$1.50 to \$4.50 | **rise of \$3.00** |
| *May* : | \$4.50 to \$3.00 | **drop of \$1.50** |
| *Jun* : | \$3.00 to \$1.00 | **drop of \$2.00** |

The greatest drop occurred during March.

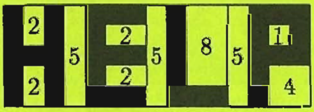
11. **(C)** Since 81 took the test, the median (middle) score is the $41^{st}$. The test interval containing the $41^{st}$ score is labeled 70.

12. **(E)** The six permutations of $+, -$ and $\times$ yield these results:

$$5\times 4+6-3=20+6-3=23$$
$$5\times 4-6+3=20-6+3=17$$
$$5+4\times 6-3=5+24-3=26$$
$$5-4\times 6+3=5-24+3=-16$$
$$5+4-6\times 3=5+4-18=-9$$
$$5-4+6\times 3=5-4+18=19.$$

The only result listed is 19.

13. **(D)** The white portion can be partitioned into rectangles as shown.




The sum of the areas of the white rectangles is $4(2)+3(5)+8+1+4=36$.

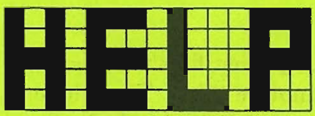
**OR**

Compute the area of the black letters and subtract it from $5\times 15=75$, the total area of the sign:

**H:** $2(1\times 5)+1\times 1=11.$
**E:** $1\times 5+3(2\times 1)=11.$
**L:** $1\times 5+1\times 2=7.$
**P:** $1\times 5+2(1\times 1)+1\times 3=10.$
The area of the white portion is $75-(11+11+7+10)=36.$

**OR**

Superimpose a $1 \times 1$ grid on the sign and count the 36 white squares:



14. **(C)** Only 3's can complete the 2 by 2 square whose diagonal is given. If two entries in a row or column are known, the third is determined. Use this to complete the table:

| | | |
|---|---|---|
| 1 | | |
| | 2 | |
| | | |

$\Rightarrow$

| | | |
|---|---|---|
| 1 | 3 | |
| 3 | 2 | |
| | | |

$\Rightarrow$

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 2 | 1 |
| 2 | 1 | |

$\Rightarrow$

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 2 | $A$=1 |
| 2 | 1 | $B$=3 |

Thus, $A + B = 4$.

15. **(A)** The sum of the four numbers is $4 \times 85 = 340$, so the sum of the remaining three numbers is $340 - 97 = 243$. Thus the mean of these three numbers is $243/3 = 81$.

16. **(C)** Using the common denominators and simplifying yield

$$\frac{1}{1+\dfrac{1}{2+\dfrac{1}{3}}} = \frac{1}{1+\dfrac{1}{\dfrac{7}{3}}} = \frac{1}{1+\dfrac{3}{7}} = \frac{1}{\dfrac{10}{7}} = \frac{7}{10}.$$

**OR**

Clearing fractions and simplifying yield

$$\frac{1}{1+\dfrac{1\times 3}{\left(2+\dfrac{1}{3}\right)\times 3}} = \frac{1}{1+\dfrac{3}{6+1}} = \frac{1\times 7}{\left(1+\dfrac{3}{7}\right)\times 7} = \frac{7}{7+3} = \frac{7}{10}.$$

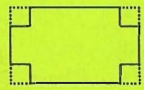
**OR**

Estimating, the result is between

$$\frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \quad \text{and} \quad \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$$

The only choice in this interval is $\frac{7}{10}$.

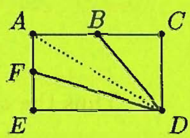
17. **(B)** The interior (or exterior) has the same surface area as one side of the sheet of cardboard after the corners have been removed. The area of the sheet is $30 \times 20 = 600$ and the area of each of the square corners removed is $5 \times 5 = 25$, so the answer is $600 - (4 \times 25) = 500$.



18. **(A)** Rectangle $ACDE$ has area $32 \times 20 = 640$. Triangle $BCD$ has area $(16 \times 20)/2 = 160$, and triangle $DEF$ has area $(10 \times 32)/2 = 160$. The remaining area, $ABDF$, is $640 - (160 + 160) = 320$.

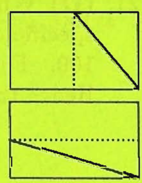
**OR**

Insert diagonal $\overline{AD}$. The areas of triangles $ABD$ and $BCD$ are $(AB)(CD)/2$ and $(BC)(CD)/2$ which are equal since $AB = BC$. Hence half the area in the rectangle above $\overline{AD}$ is in $ABDF$. Similarly, triangles $ADF$ and $DEF$ have equal areas, and half the area in the rectangle below $\overline{AD}$ is in $ABDF$. Thus, the area of $ABDF$ is $(32 \times 20)/2 = 320$.




**OR**

Draw a perpendicular from point $B$ to $\overline{ED}$ to show that the area of $\triangle BCD$ is one fourth of the area of $ACDE$. Similarly, draw a perpendicular from point $F$ to $\overline{CD}$ to show that the area of $\triangle DEF$ is one fourth of the area of $ACDE$. Thus the area of $ABDF$ is one half of the area of $ACDE$, or $(32 \times 20)/2 = 320$.



19. **(A)** Each number in the first set of numbers is 1800 more than the corresponding number in the second set:

$$\begin{array}{ccccc} 1901, & 1902, & 1903, & \ldots, & 1993 \\ \underline{-101}, & \underline{-102}, & \underline{-103}, & \ldots, & \underline{-193} \\ 1800, & 1800, & 1800, & \ldots, & 1800 \end{array}$$

Thus the sum of the first set of numbers is $93 \times 1800 = 167,400$ more than the sum of the second set.

20. **(D)** Since $10^{93} = 1\overbrace{00\cdots00}^{93\text{ zeros}}$, we have

$$\begin{array}{r} 100\cdots000 \\ \underline{-\quad 93} \\ \underbrace{99\cdots9}_{91\text{ nines}}07 \end{array}$$ and the sum of the digits is $(91 \times 9) + 7 = 826$.

**OR**

Look for a pattern using simpler cases:

$$\begin{aligned} 10^2 - 93 &= 100 - 93 &&= 07 \\ 10^3 - 93 &= 1,000 - 93 &&= \underline{9}07 \\ 10^4 - 93 &= 10,000 - 93 &&= \underline{99}07 \\ 10^5 - 93 &= 100,000 - 93 &&= \underline{999}07 \\ &\vdots && \vdots \\ 10^{93} - 93 &= &&= \underbrace{999\ldots9}_{91\text{ nines}}07 \end{aligned}$$

Thus the sum of the digits is $(91 \times 9) + 7 = 826$.

21. **(D)** When a problem indicates a general result, then it must hold for any specific case. Therefore, suppose the original rectangle is 10 by 10 with area 100. The new length is $10 + 2 = 12$ and the new width is $10 + 5 = 15$. Hence the new area is $12 \times 15 = 180$ for an increase of 80%.

**OR**

The length is changed to 120%, or 1.2 times its original value, and the width is changed to 150%, or 1.5 times its original value. Since area is length times width, the new area is $1.2 \times 1.5 = 1.8$ times the original area. Thus the area is increased by 80%.

22. **(D)** Ten 2's are needed in the unit's place in counting to 100 and ten more 2's are used in the ten's place. With the remaining two 2's he can number 102 and 112 and continue all the way to 119 before needing another 2.

23. **(C)** Since $P$, $T$ and $Q$ must finish in front of $S$, $S$ cannot be third. Since $P$ is the winner, $P$ cannot be third. Thus the only possible orders are $PRTQS$, $PTRQS$, $PTQRS$ and $PTQSR$, which show that anyone except $P$ and $S$ could finish third.

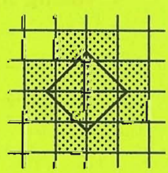
**OR**

Since $PTQS$ must finish in that order and $R$ can finish anyplace except ahead of $P$, it follows that the only possible orders are $PRTQS$, $PTRQS$, $PTQRS$, $PTQSR$. Thus $T$, $R$ and $Q$ might have finished third, but $P$ and $S$ could not have finished third.

24. **(C)** After completing some more rows, the pattern of squares as the last entry in each row becomes apparent. Thus, the line containing 142 ends in 144, and the line above it ends in 121. Therefore 121 is directly above 143, and 120 is above 142.

$$\begin{array}{ccccccccc}
 & & & \boxed{1} & & & & & \\
 & & 2 & 3 & \boxed{4} & & & & \\
 & 5 & 6 & 7 & 8 & \boxed{9} & & & \\
10 & 11 & 12 & 13 & 14 & 15 & \boxed{16} & & \\
 & & & \vdots & & & & & \\
 & & & & \cdots & 119 & 120 & \boxed{121} & \\
 & & & & \cdots & 141 & 142 & 143 & \boxed{144}
\end{array}$$

25. **(E)** Using the Pythagorean Theorem, the length of the diagonal of the card is $\sqrt{(1.5)^2+(1.5)^2} = \sqrt{4.5} \approx 2.1$. This is longer than 2, the length of two adjacent squares. The figure shows 12 squares being touched.



**Query.** Is 12 the maximum number of squares that can be touched?