AMERICAN MATHEMATICS COMPETITIONS

AJHSME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

7th ANNUAL
AMERICAN JUNIOR HIGH SCHOOL MATHEMATICS EXAMINATION
(AJHSME)

THURSDAY, NOVEMBER 21, 1991

Sponsored by

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using material normally associated with the seventh and eighth grade mathematics curriculum. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students.

Questions and comments about the problems and solutions (but not requests for the Solutions Pamphlet) should be addressed to:

Mr Bruce Brombacher, AJHSME Chairman
Jones Middle School
Upper Arlington, OH 43221

Orders for prior year Examination questions and Solutions Pamphlets or Problem Books should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0658

1. **(B)** Write the problem vertically and compute the difference:

$$\begin{array}{r} 1,000,000,000,000 \\ -\ \ 777,777,777,777 \\ \hline 222,222,222,223 \end{array}$$

OR

What must be added to 777,777,777,777 to get 1,000,000,000,000? From the right, one adds a final digit of 3 and then eleven 2's.

2. **(C)** Using the standard order of operations, first simplify the numerator and then the denominator. Finally compute the quotient:

$$\frac{16+8}{4-2} = \frac{24}{2} = 12.$$

Note. Keying $16 + 8 \div 4 - 2$ on the calculator will give an incorrect answer for this problem. The problem means $(16 + 8) \div (4 - 2) = 24 \div 2 = 12$.

3. **(E)** Using arithmetic notation

$$\begin{array}{r} 200,000 \\ \times\ 200,000 \\ \hline 40,000,000,000 \end{array}$$

OR

Using scientific notation $(2 \times 10^5)(2 \times 10^5) = 4 \times 10^{10} = 40 \times 10^9 = 40$ billion.

OR

Two hundred times two hundred is forty thousand. A thousand thousands is a million. The answer is forty thousand millions, or forty billion.

4. **(E)** Each of the five numbers on the left side of the equation is approximately equal to 1,000. Thus N can be found by computing the difference between 1,000 and each number, so $N = 9 + 7 + 5 + 3 + 1 = 25$.

OR

Since
$$\begin{aligned} & 991 + 993 + 995 + 997 + 999 \\ &= (1000-9) + (1000-7) + (1000-5) + (1000-3) + (1000-1) \\ &= 5000 - (9 + 7 + 5 + 3 + 1) = 5000 - 25, \end{aligned}$$

it follows that $N = 25$.

5. **(B)** A collection of non-overlapping dominoes must cover an even number of squares. Since checkerboard **(B)** has an odd number of squares, it follows that it cannot be covered as required. A little experimentation shows how the other checkerboards can be covered.

6. **(C)** First mark the largest number in each column.

10	6	4	3	2
11	$\boxed{7}$	14	10	8
8	3	4	5	$\boxed{9}$
$\boxed{13}$	4	$\boxed{15}$	$\boxed{12}$	1
8	2	5	9	3

Determine if any of the marked numbers is the smallest in its row. Only 7 is.

Note. One could also begin by finding the smallest number in each row. Then a check of the columns yields the answer 7.

7. **(D)** Rounding each number to one significant digit (highest place value) yields

$$\frac{(500,000)(10,000,000)+(10,000,000)(500,000)}{(20,000)(.05)}$$

which equals $\dfrac{(500,000)(10,000,000+10,000,000)}{1,000}$

which equals $(500)(20,000,000) = 10,000,000,000.$

8. **(D)** The largest quotient would be a positive number. To obtain a positive quotient either both numbers must be positive or both must be negative. Using two positive numbers, the largest quotient is $\frac{8}{1} = 8$. Using two negative numbers, the largest quotient is $\frac{-24}{-2} = 12$.

9. **(B)** A number is divisible by 3 if it is a multiple of 3, and it is divisible by 5 if it is a multiple of 5. There are 15 multiples of 3, and 9 multiples of 5 which are whole numbers less than 46. However, 3 numbers (15, 30 and 45) which are divisible by both 3 and 5 have been counted twice. Thus the total number which are divisible by either 3 or 5 or both is $15 + 9 - 3 = 21$.

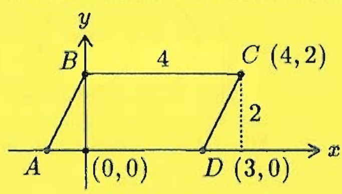
OR

Using the Sieve of Eratosthenes and marking each 3[rd] number, multiples of 3, and each 5[th] number, multiples of 5, yields

$$1, 2, \boxed{3}, 4, \boxed{5}, \boxed{6}, 7, 8, \boxed{9}, \boxed{10}, 11, \ldots, \boxed{45}, 46.$$

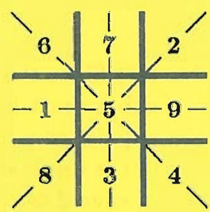
Thus, the total number which are divisible by either 3 or 5 or both is 21.

10. **(B)** The parallelogram rests on the horizontal axis. Since the coordinates of point C are $(4, 2)$, it follows that the height of the parallelogram is 2. Since point B is $(0, 2)$, it follows that the length of the base $\overline{BC}$ is 4. The area of a parallelogram is base times height. Thus the area is $4 \times 2 = 8$.



11. **(B)** After the 5 is selected, a sum of 10 is needed. There are four pairs that yield 10: 9+1, 8+2, 7+3, 6+4. Thus there are four 3-element subsets which include 5 and whose sum is 15.

Note. In the classic 3×3 "magic square" there are 4 lines through the middle of the square. The sum of the numbers along each line equals 15 and includes the number 5.



12. **(D)** Any fraction of the form $\dfrac{(k-1)+k+(k+1)}{k}$ equals 3, since $(k-1) + k + (k+1) = 3k$ and $\dfrac{3k}{k} = 3$. The denominator of the fraction must equal the middle term of the numerator. Thus $N = 1991$.

OR

Note that $\dfrac{2+3+4}{3} = \dfrac{9}{3} = \dfrac{3 \times 3}{3} = 3 \times \dfrac{3}{3} = 3 \times 1 = 3$.

Also, $\dfrac{1990 + 1991 + 1992}{N} = \dfrac{3 \times 1991}{N} = 3 \times \dfrac{1991}{N} = 3 \times 1 = 3$.

Thus N must equal 1991.

13. **(C)** Since $2 \times 5 = 10$, each zero at the end of the product comes from a product of 2 and 5 in the prime factorization of the number. Since $25 = 5 \times 5$ and $8 = 2 \times 2 \times 2$, it follows that there are fourteen factors of 5 and 9 factors of 2. This yields 9 pairs of 2×5 and results in 9 zeros at the end of the product.

OR

Multiplying the given numbers using a calculator gives an answer equivalent to 3.125×10^{12} which equals $3,125,000,000,000$. This results in 9 zeros at the end of the product.

OR

Factoring each number yields

$$\overbrace{2 \times 2 \times 2}^{8} \times \overbrace{2 \times 2 \times 2}^{8} \times \overbrace{2 \times 2 \times 2}^{8}$$
$$\times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25} \times \underbrace{5 \times 5}_{25}.$$

Pairing each factor of 2 with a factor of 5 yields $(2 \times 5)^9 \times 5^5 = 10^9 \times$ (an odd number). Thus the product ends in nine zeros.

OR

Since $25 \times 25 \times 8 = 25 \times 200 = 5000$, regrouping yields

$$(25 \times 25 \times 8) \times (25 \times 25 \times 8) \times (25 \times 25 \times 8) \times 25$$
$$= 5000 \times 5000 \times 5000 \times 25 = 125,000,000,000 \times 25 = 3,125,000,000,000.$$

14. **(D)** If one student earns $5 + 5 + 5 = 15$ points, no other student can earn more than $3 + 3 + 3 = 9$ points.
If one student earns $5 + 5 + 3 = 13$ points, no other student can earn more than $3 + 3 + 5 = 11$ points.
However, if one student earns $5 + 3 + 3 = 11$ or $5 + 5 + 1 = 11$ points, some other student can earn $3 + 5 + 5 = 13$ or $3 + 3 + 5 = 11$ points.
Thus 13 points is the smallest number of points a student must earn to be guaranteed of earning more points than any other student.

15. **(C)** When the one-foot cube is removed, three square feet of surface area are "removed", but three new square feet of surface area are "uncovered". Thus, the original surface area is unchanged.

16. **(B)** A fold from the top leaves #9-16 on the bottom.
A fold from the bottom leaves #9-12 on the bottom.
A fold from the right leaves #9 and #10 on the bottom.
A fold from the left leaves #10 on the bottom with number #9 moving to the top.

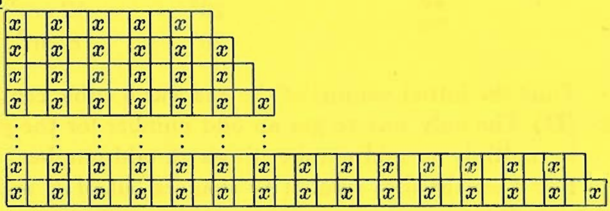
17. **(C)** The first row has 10 seats, so 5 students can sit in row 1. The second row has 11 seats, so 6 students can sit in row 2. The third row has 12 seats, so 6 students can sit in row 3. ... The last (20^{th}) row has 29 seats, so 15 students can sit in row 20. The sum is $5+6+6+7+7+\cdots+14+14+15$. Regrouping yields

$$\begin{array}{rl} & 5+\ 6+\ 6+\ 7+\ 7+\ 8+\ 8+\ 9+\ 9+10 \\ + & 15+14+14+13+13+12+12+11+11+10 \\ \hline = & 20+20+20+20+20+20+20+20+20+20 \end{array} = 10(20) = 200.$$

OR

Draw a diagram:

Row	Seats	Students
1	10	5
2	11	6
3	12	6
4	13	7
⋮	⋮	⋮
19	28	14
20	29	15



Thus, the sum is $5+6+6+7+7+\cdots+14+14+15 = 200$.

18. **(C)** Regardless of the scale on the vertical axis, 9 **X**'s out of 30 **X**'s represent employees who have worked 5 years or more. This is $\frac{9}{30}$ or 30%.

19. **(C)** Since the mean is 10, it follows that the sum of the numbers is $10 \times 10 = 100$. Taking the smallest possible values for the 9 smaller numbers would give the largest possible value of the tenth number. Thus, the largest possible number is $100-(1+2+3+4+5+6+7+8+9) = 100-45 = 55$.

20. (**A**) For the sum to be 300, $A = 2$ in the hundreds' place, since $A = 1$ gives numbers too small and $A = 3, 4, \ldots$ makes the sum too large. If $A = 2$ then $B = 7$, since $A = 2$ and $B = 8$ or 9 would be too large for the ones' place and $A = 2$ and $B = 6, 5, \ldots$ would not be enough to carry a 1 from the tens' to the hundreds' place. Thus, if $A = 2$ and $B = 7$ and $A + B + C = 10$ in the ones' place, then $C = 1$.

OR

Since the second column carry is 1, $A = 2$. The first column carry is also 1 since the sum of B and C is less than or equal to 17 and $A = 2$. In the second column, this makes $A + B +$ (first column carry) $= 2 + B + 1 = 10$, so $B = 7$. Then from the first column $A + B + C = 2 + 7 + C = 10$, so $C = 1$.

21. (**A**) The rate of change is 4 cubic centimeters per 3°. The temperature change is $32^\circ - 20^\circ = 12^\circ$. The corresponding change in volume is $12 \times \frac{4}{3} = 16$ cubic centimeters. Thus the initial volume of the gas was $24 - 16 = 8$.

OR

Work backwards, changing the volume of the gas by 4 cubic cm each time the temperature changes by 3°:

Temperature	Volume
32°	$24\ cm^3$
29°	$20\ cm^3$
26°	$16\ cm^3$
23°	$12\ cm^3$
20°	$8\ cm^3$

Thus the initial volume of the gas was 8 cubic centimeters.

22. (**D**) The only way to get an odd number for the product of two numbers is to multiply an odd number times an odd number. This happens if one spins 1 or 3 on the first spinner (2 chances out of 3) and 5 on the second spinner (1 chance out of 3). Thus, the probability of an odd product is $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$. If one does not get an odd product, then the product is even. Hence the probability of an even product is $1 - \frac{2}{9} = \frac{7}{9}$.

OR

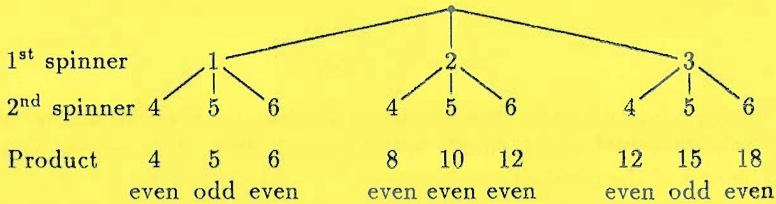
The sample space of pairs to be multiplied is:

$\boxed{1 \times 4}$	$\boxed{2 \times 4}$	$\boxed{3 \times 4}$
1×5	$\boxed{2 \times 5}$	3×5
$\boxed{1 \times 6}$	$\boxed{2 \times 6}$	$\boxed{3 \times 6}$

Successful pairs are marked. The probability is $\frac{7}{9}$.

OR

Use a tree diagram:



Thus the probability that the product is even is $\frac{7}{9}$.

23. (A) There are 100 females in the band, 80 in the orchestra, and 60 in both. Thus, there are $(100 + 80) - 60 = 120$ females in at least one of the groups. Since the total is 230, then there are $230 - 120 = 110$ males in at least one of the groups. There are 80 males in band and 100 males in orchestra, thus to find the number of males in both, $(80 + 100) - ? = 110$. There are 70 in both. Finally, the number of males in band who are not in orchestra is $80 - 70 = 10$.

OR

Make a chart for the given information:

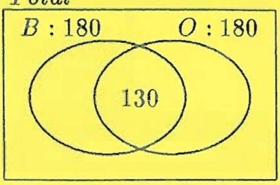
	# in band	#in orchestra	# in both
Male :	80	100	?
Female :	100	80	60
Totals :	180	180	60+?

The total number of students is $180 + 180 - (60 + ?) = 230.$
Solving the equation we obtain $180 + 180 - 60 - ? = 230,$
$? = 70.$

Since 70 males are in both band and orchestra, it follows that $80 - 70 = 10$ males are in band who are not in orchestra.

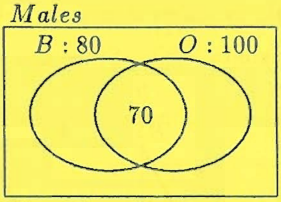
OR

Total



$180+180-230 = 130$ *in both*
$130-60 = 70$ *males in both*

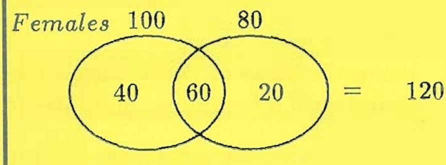
Males

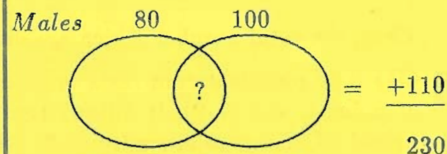


$80-70 = 10$ *males in band only*

OR

Band **Orch**





There are 130 in both, thus ? equals 70. It follows that 10 males are in band only.

24. **(E)** Since the edge of each smaller cube must be a whole number, the smaller cubes must be $1 \times 1 \times 1$ or $2 \times 2 \times 2$ cubes. There can be only one smaller cube of edge 2, so the rest of the smaller cubes have edge 1. Since the volume of the original cube was $3 \times 3 \times 3 = 27$ cubic cm, and the volume of the cube of edge 2 is $2 \times 2 \times 2 = 8$ cubic cm, then there must be $27 - 8 = 19$ cubes of edge 1 (volume = 1 cubic cm). There are a total of 20 cubes so $N = 20$.

25. **(C)** Since 1/4 turns white, it follows that 3/4 remains black after the first change. After the second change, 3/4 of the remaining 3/4 stays black, or $\frac{3}{4} \times \frac{3}{4}$ remains black. Thus, after the fifth change, the amount remaining black is

$$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{243}{1024}$$

of the original area.